SINGLE AND MULTI-PERSON CONTROLLED DIFFUSIONS

BY

STANLEY ROY PLISKA

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This paper is concerned with the optimal control of a one-dimensional stationary diffusion process on a compact interval, a concept developed primarily by P. Mandl (Analytical Treatment of One-Dimensional Markov Processes, New York: Springer-Verlag, 1968) The drift and diffusion coefficients depend upon a stationary control assumed to be a piece-wise continuous function of the state. The costs generated by the process are functions of both the control and the sample path of the process. Mandl's concept of a controlled diffusion process is generalized by allowing the controls to be vector-valued with the set of admissible control actions defined by a piecewise continuous set-valued function on the state space. Both single and multi-person problems are considered. The main results include necessary and sufficient conditions for a control to be "optimal" and conditions assuring the existence of a piece-wise continuous optimal control. Applications are given to problems of controlling reservoirs, pollution, queues, investments, welfare, and warfare.

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CHAPTER I

INTRODUCTION AND SUMMARY

This dissertation is concerned with the control of a one-dimensional diffusion process. The concept of a controlled diffusion process used here is almost identical to that defined by Mandl [16]. The system we both consider is a stationary diffusion process defined on a compact interval of the real line. The drift and diffusion coefficients depend upon a stationary control which is state dependent. The costs generated by the process are functions of both the control and the sample path of the process. Mandl's concept of a controlled diffusion process is generalized by allowing controls to be vector-valued with the set of admissible control actions defined by a piecewise continuous set-valued function on the state space.

If the process is controlled by a single person and the process generates a single stream of costs, then we have the ordinary optimal control problem of Chapter II. These problems involve finding controls which minimize expected costs. In the case of undiscounted costs, Mandl's results are generalized to account for our previously mentioned restriction on admissible control functions. With discounting, the minimal expected discounted cost as a function of the initial state of the process is shown to be the unique solution of a differential equation. A necessary and sufficient condition is given for a control to yield the minimal expected discounted cost, and a method is presented for computing this optimal control

In Chapter III, we suppose the process is controlled by two persons, one player who chooses one component of the control wants to minimize the expected cost, while the second player who chooses the other control component wants to maximize the expected cost. The problem of controlling this process is a sequential, zero sum, two-person game. Moreover, by an optimal control is meant a saddlepoint of the expected cost function. The expected cost associated with a saddlepoint is called the value of the game. The value, if it exists, is shown to be the unique solution of a differential equation. Furthermore, a necessary and sufficient condition is given for a control to be optimal.

If the diffusion process is controlled by N persons and it generates N streams of costs, then the problem of controlling the process bcomes a sequential, non-zero sum, N-person game. Suppose the i^{th} player, who operates the i^{th} control component, wants to minimize the expected costs of the i^{th} cost stream. Then by an optimal control will be meant one that is a Nash equilibrium point of the expected costs corresponding to all admissible controls. In Chapter III each Nash equilibrium point is shown to be the unique solution of a differential equation, and a necessary and sufficient condition is given for a control to be optimal

A piecewise continuous optimal control may not exist for a problem of any type mentioned above. In Chapter II, a piecewise continuous optimal control is shown to exist for the single person control problem under a variety of alternative conditions. One is that the action set in each state is finite and state independent, and the drift, diffusion, and continuous movement costs are analytic in the state A second is that the action set in each state be convex the diffusion coefficient be action independent, the drift coefficient be affine in the action, and the continuous movement cost be strictly convex in the action Conditions of the latter type are also given for multiperson control problems.

Chapters II and III provide several applications of the models developed herein The ordinary optimal control model is applied to the problems of controlling a reservoir, controlling pollution, optimizing a queueing system, and making optimal investments The zero sum, two-person game model is applied to the problem of determining how many people should be receiving welfare The non-zero sum N-person game model is applied to the problems of pollution and warfare Chapters II and III also include numerous example calculations of the optimal control for each type of model; some of these examples pertain to the applications that are discussed

CHAPTER II

SINGLE PERSON CONTROLLED DIFFUSIONS

This chapter describes a class of single person controlled one-dimensional diffusion processes where the control is a vector-valued function on the state space. These processes generate costs, and the optimal control problem is to choose an admissible control that minimizes expected costs The following three sections provide results for discounted costs, undiscounted costs with a non-conservative process, and undiscounted costs with a conservative process Generalizing Mandl [16], the major results in these sections are necessary and sufficient conditions for a control to be optimal as well as characterizations of the expected costs corresponding to an optimal control The method for solving a problem is basically the same in each case A differential equation is solved and the solution is used to determine the optimal control.

Section 4 establishes sufficient conditions for the existence of piecewise continuous optimal controls The remaining four sections discuss four potential applications of single person controlled diffusion processes Each such section describes how these processes can be used as models of the physical systems being considered, and then examples are provided to demonstrate how actual problems could be solved

1 The Controlled Diffusion Process.

Consider a diffusion process with state space S, a compact interval $[r_0, r_1]$ of the real line E, which is controlled by a single person. For some positive integer n and some compact set $K \subset E^n$, the control is a vector-valued function on S with range K. Let A_s be a point-to-set map from S into K such that A_s is piecewise continuous in the Hausdorff metric and for each $s \in S$ the set A_s is a non-empty compact subset of K. Each time the process is observed in state s, an action "a" is chosen from the set A_s. The set M of admissible controls consists of all piecewise continuous functions $a(\ )$ on S with range in E^n such that the action $a(s) \in A_s$ for each $s \in S$. A function $a(\ )$ is an admissible control if and only if $a(\ ) \in M$. Throughout this chapter it should be clear from the context whether the letter a denotes an admissible control $a = a(\ ) \in M$ or an admissible action $a \in A_s$ for some $s \in S$. In the sequel we shall assume A_s is such that M is non-void. Although this is always true whenever $K \subset E$, it is not known whether in general $M \neq \phi$ if $K \subset E^n$, $n \geq 2$.

In order to characterize the map A_s, let the set Z consist of all compact subsets of K. We define a metric ρ on Z as follows. For any $a \in E^n$ and $Y \in Z$ let $D(a,Y)$ be the Euclidean distance between the point a and the set Y. For any $Y_1, Y_2 \in Z$ let

$$\rho(Y_1, Y_2) = \sup_{a \in Y_1} D(a, Y_2) + \sup_{a \in Y_2} D(a, Y_1)$$

Then (Z, ρ) is a metric space with the Hausdorff metric, and the map

A_s from S into (Z,ρ) is continuous at s for all but a finite number of $s \in S$.

It can be shown (for example, Hogan [12]) that A_s is continuous in the Hausdorff metric at the point s^∞ if and only if A_s is both upper semi-continuous, that is, a closed map, and lower semi-continuous at s^∞. The map A_s is upper semi-continuous at s^∞ if (i) $s^i \to s^\infty$, (ii) $a^i \to a^\infty$, and (iii) $a^i \in A_{s^i}$ together imply $a^\infty \in A_{s^\infty}$. The map A_s is lower semi-continuous at s^∞ if (i) $s^i \to s^\infty$ and (ii) $a^\infty \in A_{s^\infty}$ together imply there exists a sequence of actions $a^i \in A_{s^i}$ such that $a^i \to a^\infty$.

The definition of a controlled diffusion process is a slight generalization of Mandl's [16, p. 157]. Let $d(s,a)$ be a continuous positive real-valued function on $S \times K$. Then for $a(\cdot) \in M$ the piecewise continuous function $d(s,a(s))$ is the diffusion coefficient of the process. Similarly, let $b(s,a)$ be a continuous real-valued function on $S \times K$ so that $b(s,a(s))$ is the drift coefficient of the diffusion process.

Following Mandl, with a given control $a(\cdot) \in M$ the diffusion process is completely specified by the generalized classical differential operator

$$D = d(s,a(s))\frac{d^2}{ds^2} + b(s,a(s))\frac{d}{ds}$$

together with Feller's [7,9] boundary condition

$$\kappa_j v(r_j) + \theta_j \left(v(r_j) - \int_S v(s)\, d\mu_j(s) \right) - (-1)^j \pi_j v'(r_j)$$

$$+ \sigma_j (Dv)(r_j) = 0 , \qquad j = 0,1,$$

where $v(s)$ is some function whose second derivative is piecewise continuous on S. At each boundary r_0, r_1 the four non-negative parameters κ_j, σ_j, π_j and θ_j, at least one of which must be positive, correspond respectively to the phenomena of absorption, adhesion, reflection and instantaneous return. Corresponding to θ_j is the probability distribution function $\mu_j(s)$ where $\int_{(r_0, r_1)} d\mu_j(s) = 1$.

Feller [8] and Ito and McKean [14] present partial probabilistic interpretations of these boundary conditions in the case of diffusion processes, and Ito and McKean [13] present a complete description in the case of Brownian motion. Their results are briefly described here.

The reflecting barrier process with $\kappa_j = \sigma_j = \theta_j = 0$ for $j = 0,1$ can be described by constructing a diffusion process on E. With $L = r_1 - r_0$ define the point-to-set map $f: S \to E$ as

$$f(s) = \{x \in E \mid x = s + 2nL \text{ or } x = 2r_0 - s + 2nL \text{ for some } n = 0, \pm 1, \pm 2, \ldots\}$$

Note that $\bigcup_{s \in S} f(s) = E$ and the inverse $f^{-1}(x) = \{s \in S : x \in f(s)\}$ equals a unique $s \in S$ for each x [illegible]. For ea[illegible] define the dri[illegible] and diffusion coefficients of the constructed process to be [illegible] equal to the drift and diffusion coefficients, respectively, of the reflecting barrier process at the point $f^{-1}(x)$. If the constructed process is represented by the sample path $t \to x(t)$ then the sample path $s^+(t) = f^{-1}(x(t))$ represents the reflecting barrier process.

A diffusion process $s(t)$ without reflecting barriers $(\pi_0 = \pi_1 = 0)$ behaves like the reflecting barrier process $s^+(t)$ up to the first passage time $m = \min\{t \mid s^+(t) = r_0 \text{ or } s^+(t) = r_1\}$. Then, if $s^+(m) = r_j$, $s(t) = r_j$ for an exponential holding time e_j with conditional law

$$P(e_j > \tau \mid s^+) = e^{-(\kappa_j+\theta_j)\tau/\delta_j} .$$

At time $m + e_j$ either the process terminates (absorption) with probability $\kappa_j/(\theta_j + \kappa_j)$ or it starts afresh by jumping to the point $\rho \in (r_0, r_1)$ with conditional law

$$P(s(m + e_j) = \rho \mid e_j, s^+) = \theta_j \nu_j(d\rho)/(\theta_j + \kappa_j) .$$

The interpretation of the boundary condition of reflection combined with absorption and/or adhesion $(\pi_j(\kappa_j + \delta_j) > 0, \theta_j = 0)$ is rather more complicated. Briefly, with $\delta_j > 0$ the process behaves like the reflecting barrier process with a stochastic time scale change that counts standard time while $s^+(t) \neq r_j$ but runs slow on the barrier with the result that, compared to the reflecting barrier process, this process lingers at the boundary longer than it should. With $\kappa_j > 0$ the process behaves as if $\kappa_j = 0$ until it is killed on the boundary r_j at a random time that is a function of the visiting set $\{t \mid s(t) = r_j\}$. If $\eta_j(\kappa_j + \sigma_j) > 0$ and the passage time $\tilde{m}(\tau) = \inf\{t > \tau \mid s(t) \neq r_j\}$ then the conditional probability

$$P(\tilde{m}(\cdot) = 0 \mid s(\tau) = r_j) = 1 .$$

The process with both reflection and instantaneous return occurring at a boundary $(\eta_j \theta_j > 0)$ can be constructed from a process with both reflection and absorption occurring at the boundary $(\eta_j \kappa_j > 0)$ as was done by Mandl [16, pp. 64-66].

The diffusion process generates costs according to its sample path and control (Mandl [16, p. 148]). These costs are of three types.

The continuous movement cost is the cost rate per unit time. Let $c(s,a)$ be a continuous function from $S \times K$ into E. If $s(t)$ is the sample path of the process and $a(s)$ the control then the integral of $c(s(t),a(s(t)))$ over a time interval equals the total continuous movement cost generated over this time interval.

The second kind of cost is associated with jumps (instantaneous returns) by the process from the boundaries. For $j = 0,1$ let $v_j(s)$ be a function from S into E which is integrable with respect to $\mu_j(s)$. If the process jumps from boundary r_j to the point $s \in (r_0, r_1)$ at time t then there arises at this time the jump cost $v_j(s)$. Denote by $\phi_j(t,s)$, $t \geq 0$, $s \in S$, $j = 0,1$ the integer-valued random variable representing the number of jumps made by the process up through time t from boundary r_j into the interval $[r_0, s]$. Then the total cost due to jumps from r_j up through time t equals the integral over $[r_0, r_1]$ of $\nu_j(s)\phi_j(t,ds)$.

The third kind of cost depends upon the termination of the process. If the process is absorbed at boundary r_j, then at this termination time there arises the cost λ_j, $j = 0,1$.

Following Mandl [16, p. 149], if $C(t)$ is the total of the costs generated by the process up through time t, then the Laplace-Stieljes transform

$$\int_0^\infty e^{- t} dC(t)$$

can be regarded as the total, discounted, infinite horizon cost generated by the process, where the discount factor is $e^{-\lambda}$ and $\lambda > 0$. Given a controlled diffusion process, admissible control, and discount factor, let $v(s)$ denote the conditional expectation of the discounted cost of this process given its initial state s, that is

$$v(s) = E_s \int_0^\infty e^{-\lambda t} dC(t)$$

Mandl [16, p. 149] proves the following result.

Theorem 1. The expected discounted cost $v(s)$ corresponding to $a(\cdot) \in M$ is the unique function on S such that $v'(s)$ is continuous,

(1) $d(s,a(s))v''(s) + b(s,a(s))v'(s) - \lambda v(s) + c(s,a(s)) = 0$

holds for every $s \in (r_0, r_1)$ which is a continuity point of $a(s)$, and

$$(2)\quad (\theta_j + \kappa_j)v(r_j) - \theta_j \int_S (v(s) + \nu_j(s))d\mu_j(s) - (-1)^j \pi_j v'(r_j)$$

$$+ \sigma_j(\lambda v(r_j) - c(r_j, a(r_j))) - \kappa_j \lambda_j = 0, \quad j = 0,1 .$$

If the process is non-conservative and neither boundary is purely adjesive, that is,

$$\kappa_0 + \kappa_1 > 0 , \quad \kappa_j + \pi_j + \theta_j > 0 , \quad j = 0,1,$$

then Mandl [16, p. 152] shows that the expected total undiscounted cost $v(s) = E_s C(\infty)$ is finite and is the unique solution of (1) and (2) for $\lambda = 0$.

If the process is conservative $(\kappa_0 + \kappa_1 = 0)$, then the total undiscounted cost may be infinite. The number Θ in the following theorem by Mandl [16, pp. 152-157, 168] can be interpreted as the mean cost per unit time.

Theorem 2. Let $\kappa_0 = \kappa_1 = 0$ and assume at least one boundary is not purely adhesive, that is, $\pi_0 + \theta_0 + \pi_1 + \theta_1 > 0$. If $v(s,\lambda)$ is the expected discounted cost corresponding to $\lambda > 0$ and some $a(\cdot) \in M$, then

$$\lim_{\lambda \downarrow 0} \lambda v(s,\lambda) = \Theta \quad \text{and} \quad \lim_{\lambda \downarrow 0} \frac{d}{ds} v(s,\lambda) = w(s) ,$$

where Θ is some number independent of the state s, and $w(s)$ is some absolutely continuous function on S. Moreover,

$$P(\lim_{t\to\infty} t^{-1}C(t) = \Theta) = 1$$

and (Θ, w) is the unique pair satisfying

$$\text{(3)} \quad d(s,a(s))w'(s) + b(s,a(s))w(s) - \Theta + c(s,a(s)) = 0$$

for every $s \in (r_0, r_1)$ which is a continuity point of $a(s)$, and

$$\text{(4)} \qquad \theta_j \int_S \left\{ \int_{r_j}^{s} w(y)dy + \nu_j(s) \right\} d\mu_j(s) + (-1)^j \gamma_j w(r_j)$$

$$+ \sigma_j (c(r_j, a(r_j)) - \Theta) = 0 \qquad j = 0,1 .$$

2. The Discounted Cost Case.

Let $v(s,a) = v(s)$ denote the expected discounted cost of a process corresponding to the admissible control $a \in M$. Then $v(s,a)$ will be the unique solution of (1), (2). The minimal expected discounted cost $v(s)$ is defined to be

$$\hat{v}(s) = \inf_{a \in M} v(s,a) .$$

An admissible control $\hat{a} \in M$ is said to be an optimal control if $v(s,\hat{a}) = \hat{v}(s)$ for all $s \in S$. The results here for the discounted cost case generalize Mandl's [16, pp. 158-173] results for undiscounted costs. The main results of this section are Theorem 3, which characterizes the minimal expected cost, and Theorem 6, which provides necessary and sufficient conditions for an admissible control to be optimal

<u>Theorem 3</u>. The minimal expected discounted cost $\hat{v}(s)$ is the unique solution of the equation

$$(5) \qquad \hat{v}''(s) + \min_{a \in A_s} \{d(s,a)^{-1}[b(s,a)\hat{v}'(s) - \lambda \hat{v}(s) + c(s,a)]\} = 0$$

satisfying

$$(6) \qquad (\theta_j + \kappa_j)\hat{v}(r_j) - \theta_j \int_S (\hat{v}(s) + \nu_j(s))d\mu_j(s) = (-1)^j \pi_j \hat{v}'(r_j)$$

$$\theta_j(\alpha_j v(r_j) - \gamma_j) - \kappa_j \lambda_j = 0 , \qquad j = 0,1,$$

where $$\gamma_j = \min_{a \in A_{r_j}} c(r_j,a) , \qquad j = 0,1 .$$

Two preliminary lemmas will be provided before the proof of Theorem 3 is presented. The first lemma, stated without proof, is a slight modification of a selection theorem due to Dubins and Savage [5, Chap. 2.16] (see also Maitra [15]). I am grateful to Robert Rosenthal for suggesting the appropriateness of Lusin's Theorem in the proof of Lemma 5.

Lemma 4. If $h(s,a)$ is a continuous real-valued function on $S \times K$, then there exists a Borel measurable function $f(s)$ on S into K such that $h(s,f(s)) = \min_{a \in A_s} h(s,a)$ and $f(s) \in A_s$ for each $s \in S$.

Lemma 5 Suppose $f(\ )$ is a function on S into K which is measurable with respect to Borel measure μ and satisfies $f(s) \in A_s$ for each $s \in S$. Then for every $\varepsilon > 0$ there exists a measurable subset $\bar{S} \subset S$ and an admissible control $a(\cdot) \in M$ such that $\mu(S - \bar{S}) < \varepsilon$ and $a(s) = f(s)$ for all $s \in \bar{S}$.

Proof: Lusin's Theorem (e.g. Royden [18]) is valid for vector-valued measurable functions, so for every $\varepsilon > 0$ there exists a measurable subset $S' \subset S$ and a continuous function $g(s)$ on S such that $\mu(S - S') < \frac{\varepsilon}{3}$ and $f(s) = g(s)$ for all $s \in S'$. It remains to show that $g(s)$ coincides with some admissible control on a large enough subset of S'.

Let $S'' = \{s \in S \mid g(s) \in A_s\}$ so clearly $S' \subset S''$ and $\mu(S - S'') < \frac{\varepsilon}{3}$. Since $g(s)$ and A_s both have closed graphs, S'' is closed. Thus there exists a sequence $\{S_i\}$ of disjoint closed intervals such that $\bigcup_{i=1}^{\infty} S_i = S''$. Without loss of generality, assume $\mu S_1 \geq \mu S_2 \geq \cdots$, so that for some integer N, $\mu(S - \bigcup_{i=1}^{N} S_i) < \frac{\varepsilon}{3}$. Note that there exists an admissible control $a(\cdot) \in M$ such that $a(s) = g(s)$ for all $s \in \bigcup_{i=1}^{N} S_i$. If we set $\overline{S} = S' \cap (\bigcup_{i=1}^{N} S_i)$, then

$$\mu(S - \overline{S}) \leq \mu(S - S') + \mu(S - \bigcup_{i=1}^{N} S_i) = \mu(S - S') + \mu(S - S'') + \mu(S'' - \bigcup_{i=1}^{N} S_i)$$

$< \varepsilon$, so the pair $(a(s), \overline{S})$ is as desired.

<u>Proof of Theorem 3</u>. Equation (5) has a unique solution $v(s)$ satisfying boundary conditions (6) by Theorem 3 of Chapter III. The remainder of this proof will be in two parts; first it will be shown that $v(s) \leq v(s,a)$ for all $a \in M$.

For arbitrary $a \in M$ denote $\tilde{v}(s) = v(s,a) - v(s,a)$ and define $\psi(s)$ by

$$(7) \qquad \psi(s) = \tilde{v}''(s) + d(s,a(s))^{-1}[b(s,a(s))\tilde{v}'(s) - \lambda\tilde{v}(s)]$$

$$= v''(s,a) + d(s,a(s))^{-1}[b(s,a(s))v'(s,a) - \lambda v(s,a) + c(s,a(s))]$$

$$- \{\hat{v}''(s) + d(s,a(s))^{-1}[b(s,a(s))v'(s) - \lambda v(s) + c(s,a(s))]\}$$

$$\leq v''(s,a) + d(s,a(s))^{-1}[b(s,a(s))v'(s,a) - \lambda v(s,a) + c(s,a(s))]$$

$$- \{v''(s) + \min_{a \in A_s} \{d(s,a)^{-1}[b(s,a)v'(s) - \lambda v(s) + c(s,a)]\}\} = 0$$

The last equality follows from equations (1) and (5), note by the first equality that $\psi(s)$ is piecewise continuous. Subtracting (6) from (2), it can be seen that $v(s)$ also satisfies

$$(8) \qquad (\theta_j + \kappa_j)\hat{v}(r_j) - \theta_j \int_S \hat{v}(s)d\mu_j(s) - (-1)^j \tau_j \hat{v}'(r_j)$$

$$+ \sigma_j(\lambda\hat{v}(r_j) - (c(r_j,a(r_j)) - \gamma_j)) = 0\,, \qquad j = 0,1\,.$$

Conclude from (7) and (8) that $\tilde{v}(s)$ is the expected discounted cost corresponding to the controlled diffusion process with admissible control $a(s)$, continuous movement cost $\tilde{c}(s,a(s)) = -d(s,a(a))\psi(s) \geq 0$, instantaneous return cost equal to zero, absorption cost equal to zero, and $\tilde{c}(r_j,a(r_j)) = c(r_j,a(r_j)) - \gamma_j \geq 0$, $j = 0,1$. Since all costs are non-negative it is apparent that $\tilde{v}(s) = v(s,a) - \hat{v}(s) \geq 0$ for all $s \in S$.

The final part of this proof is to show $\hat{v}(s) = \inf_{a \in M} v(s,a)$ by demonstrating the existence of a sequence $\{a_n(\cdot)\}$ of admissible controls with the property that $v(s,a_n) \to \hat{v}(s)$ as $n \to \infty$ for all $s \in S$. By Lemma 4 there exists a Borel measurable function $f(\cdot)$ from S into K such that $f(s) \in A_s$ and

$$d(s,f(s))^{-1}[b(s,f(s))\hat{v}'(s) - \lambda\hat{v}(s) + c(s,f(s))]$$

$$= \min_{a \in A_s} d(s,a)^{-1}[b(s,a)\hat{v}'(s) - \lambda\hat{v}(s) + c(s,a)]$$

for each $s \in S$. By Lemma 5 there exists a sequence of admissible controls that converges in measure to $f(s)$. Now some subsequence must converge almost everywhere to $f(s)$, so there exists a sequence $\{a_n(\cdot)\}$ of admissible controls that converges a. e. to $f(\ )$. Also, we can assume $a_n(r_j) \in A_{r_j}$ is such that $\sigma_j(c(r_j, a_n(r_j)) - \gamma_j) = 0$ for $n = 1,2,\cdots$ and $j = 0,1$.

Denote $\tilde{v}_n(s) = v(s,a_n) - \hat{v}(s)$ and define $\psi_n(s)$ as in (7) by

$$\psi_n(s) = \tilde{v}_n''(s) + d(s,a_n(s))^{-1}[b(s,a_n(s))\tilde{v}_n'(s) - \lambda v_n(s)]$$

$$= -\{\hat{v}''(s) + d(s,a_n(s))^{-1}[b(s,a_n(s))\hat{v}'(s) - \lambda v(s) + c(s,a_n(s))]\}$$

Note that the piecewise continuous function $-\psi_n(s)$ converges almost everywhere to

$$\hat{v}''(s) + d(s,f(s))^{-1}[b(s,f(s))\hat{v}'(s) - \lambda v(s) + c(s,f(s))]$$

$$= \hat{v}''(s) + \min_{a\in A_s} \{d(s,a)^{-1}[b(s,a)\hat{v}'(s) - \lambda v(s) + c(s,a)]\} = 0$$

As in (8) we see that $\tilde{v}_n(s)$ must satisfy

$$(\theta_j + \kappa_j)v_n(r_j) - \theta_j \int_S \tilde{v}_n(s)d\mu_j(s) - (-1)^j \pi_j \hat{v}'(r_j)$$
$$+ c_j \lambda v(r_j) = 0 ; \qquad j = 0,1$$

Thus $\tilde{v}_n(s)$ is the expected discounted cost of a controlled diffusion process with control $a_n \in M$, zero jump, stopping and adhesion costs, and continuous movement cost* $-d(s,a_n(s))\psi_n(s)$ Since the only cost

*Note that the continuous movement cost here is no longer the composition of a continuous function on $S \times K$ with a piecewise continuous control. However, in view of Mandl [16, pp 148-49], this presents no problem since $-d(s,a_n(s))\psi_n(s)$ is piecewise continuous on S

of the process converges a. e. to zero, we must have $\hat{v}_n(s) = v(s,a_n) - \hat{v}(s) \to 0$ as $n \to \infty$, and Theorem 3 is proved.

Theorem 6. Let $\hat{v}(s)$ be the minimal expected discounted cost for a controlled diffusion process. A control $a \in M$ is optimal if and only if

$$(9) \qquad d(s,a(s))^{-1}[b(s,a(s))\hat{v}'(s) - \lambda\hat{v}(s) + c(s,a(s))]$$

$$= \min_{a\in A_s} \{d(s,a)^{-1}[b(s,a)\hat{v}'(s) - \lambda\hat{v}(s) + c(s,a)]\}$$

for every $s \in S$ which is a continuity point of $a(\cdot)$ and

$$(10) \qquad \sigma_j(c(r_j,a(r_j)) - \gamma_j) = 0 \qquad \text{for } j = 0,1$$

Proof. Suppose (9) and (10) hold. By (5) we have

$$\hat{v}''(s) + d(s,a(s))^{-1}[b(s,a(s))\hat{v}'(s) - \lambda\hat{v}(s) + c(s,a(s))] = 0$$

and by (6) we have

$$(\theta_j + \kappa_j)\hat{v}(r_j) - \theta_j \int_S (\hat{v}(s) + \nu_j(s))d\mu_j(s) - (-1)^j \tau_j \hat{v}'(r_j)$$

$$+ \sigma_j(\lambda\hat{v}(r_j) - c(r_j,a(r_j))) - \kappa_j\lambda_j = 0, \quad j = 0,1$$

Hence $\hat{v}(s)$ satisfies (1) and (2) so $\hat{v}(s) = v(s,a)$ and $a(s)$ is an optimal control.

Conversely, suppose $a \in M$ is an optimal control but that (9) does not hold, that is, for some $\bar{s} \in S$ which is a continuity point of $a(s)$ we have

$$d(\bar{s},a(\bar{s}))^{-1}[b(\bar{s},a(\bar{s}))\hat{v}'(\bar{s}) - \lambda\hat{v}(\bar{s}) + c(s,a(\bar{s}))]$$

$$> \min_{a \in A_{\bar{s}}} \{d(\bar{s},a)^{-1}[b(\bar{s},a)\hat{v}'(\bar{s}) - \lambda\hat{v}(\bar{s}) + c(\bar{s},a)]\}$$

Defining $\psi(s)$ as in (7) we note that $\psi(s) < 0$ in some neighborhood of $\bar{s}$. Using the arguments following (7) and (8), we conclude $v(\bar{s},a) > \hat{v}(\bar{s})$, which is a contradiction.

Finally, suppose $a \in M$ is an optimal control and (9) holds, but (10) does not. Using the arguments following (7) and (8) again we have that $v(s,a) - \hat{v}(s)$ is the expected discounted cost of a process with zero continuous movement, instantaneous return and absorption costs but with positive adhesion costs. Thus $v(r_j,a) > \hat{v}(r_j)$ for $j = 0$ and/or $j = 1$, a contradiction, and Theorem 6 is proved

The minimal expected discounted cost and an optimal control may in principle be calculated for a process as follows. Define the function $f(s,y,z)$ from $S \times E^2$ into K such that $f(s,y,z) \in A_s$,

$$\sigma_j(c(r_j,f(r_j,y,z)) - \gamma_j) = 0, \quad \text{for } j = 0,1 \quad \text{and}$$

$$d(s,f(s,y,z))^{-1}[b(s,f(s,y,z))z - \lambda y + c(s,f(s,y,z))]$$

$$= \min_{a \in A_s} \{d(s,a)^{-1}[b(s,a)z - \lambda y + c(s,a)]\}$$

for all appropriate $s \in S$ and $y, z \in E$. Then the minimal expected discounted cost $\vartheta(s)$ will be the unique solution on S to

$$d(s,f(s,\vartheta(s),\vartheta'(s))\vartheta''(s) + b(s,f(s,\vartheta(s),\vartheta'(s)))\vartheta'(s)$$

$$- \lambda\vartheta(s) + c(s,f(s,\vartheta(s),\vartheta'(s))) = 0$$

satisfying (6). The function $f(s,v(s),v'(s))$ on S will then be an optimal control provided it is admissible, that is, piecewise continuous.

The following example demonstrates that the optimal control is a function of the discount factor λ.

Example.

$0 < r_0 < r_1$,

$A_s = \{a \in E \mid ks \geq |a|\}$, $k > 0$,

$d(s,a) = d > 0$,

$b(s,a) = ba/s$, $b > 0$

$c(s,a) = c > 0$,

r_0 boundary condition: $v'(r_0) = 0$ (reflection),

r_1 boundary condition: $v(r_1) = \lambda_1$ (absorption with cost λ_1).

Upon substitution into (5) one observes that the optimal action assumes either the maximum or minimum value as the derivative of the minimal expected discounted cost is respectively negative or positive. If $\dot{v}(r_0) = c/\lambda$, then the unique solution of (5) is $\hat{v}(s) = c/\lambda$. If $\hat{v}(r_0) > c/\lambda$ and $a(s) = -ks$, then $\hat{v}'(s) \geq 0$ so $a(s) = -ks$ is optimal and $v(r_1) \geq v(r_0) > c/\lambda$. Similarly, $v(r_0) < c/\lambda$ implies $a(s) = ks$ is optimal and $v(r_1) \leq v(r_0)$. Since $v(r_1) = \lambda_1$ determines $v(r_0)$ uniquely, we conclude that the optimal control is given by

$$a(s) = \begin{cases} -ks\,, & \text{if } \lambda_1 \geq c/\lambda\,, \\ ks\,, & \text{if } \lambda_1 \leq c/\lambda\,. \end{cases}$$

3. The Undiscounted Cost Case.

Mandl [16, pp. 158-173] provides results for the undiscounted cost case $(\lambda = 0)$ when the controls are real-valued functions and the sets of admissible actions are independent of the state space, that is, for some compact $K \subset E$, $A_s = K$ for all $s \in S$. The purpose of this section is to generalize his results in accordance with the formulation of section 1.

For the undiscounted cost case there are two situations: either the process is conservative or non-conservative. For the purposes of this section, the boundary conditions are said to be non-conservative if at least one boundary is absorbing and neither boundary is purely adhesive, that is

$$\kappa_0 + \kappa_1 > 0 , \qquad \kappa_j + \pi_j + \theta_j > 0 , \qquad j = 0,1 .$$

Let $v(s,a)$ denote the expected cost of such a process corresponding to the control $a \in M$. Then $v(s,a)$ will be the unique solution of (1) and (2) with $\lambda = 0$. The minimal expected cost $\hat{v}(s)$ is defined to be

$$\hat{v}(s) = \inf_{a \in M} v(s,a) .$$

An admissible control $a \in M$ is said to be optimal if $\hat{v}(s) = v(s,a)$ for all $s \in S$. The main result for non-conservative processes and undiscounted costs is the following.

Theorem 7. Suppose the boundary conditions are non-conservative. Then the minimal expected cost $\hat{v}(s)$ is the unique solution of

$$\text{(11)} \qquad \hat{v}''(s) + \min_{a \in A_s} \{ d(s,a)^{-1} [b(s,a)\hat{v}'(s) + c(s,a)] \} = 0$$

satisfying

$$\text{(12)} \qquad (\theta_j + \kappa_j)\hat{v}(r_j) - \theta_j \int_S (\hat{v}(s) + \gamma_j(s)) d\mu_j(s) - (-1)^j \pi_j \hat{v}'(r_j)$$

$$- \sigma_j \gamma_j - \kappa_j \lambda_j = 0 , \qquad j = 0,1,$$

where $\gamma_j = \min\limits_{a \in A_{r_j}} c(r_j, a)$, $j = 0,1$.

A control $a \in M$ is optimal if and only if

$$(13) \qquad d(s,a(s))^{-1}[b(s,a(s))\dot{v}'(s) + c(s,a(s))]$$

$$= \min_{a \in A_s} \{d(s,a)^{-1}[b(s,a)\dot{v}'(s) + c(s,a)]\}$$

for every $s \in S$ which is a continuity point of $a(\ )$ and

$$(14) \qquad \sigma_j(c(r_j,a(r_j)) - \gamma_j) = 0 , \qquad j = 0,1$$

<u>Proof</u> There exists a unique solution $\dot{v}(s)$ to (11), (12) by virtue of Theorem 14, Chapter III. The remainder of the proof proceeds as with Theorems 3 and 6, so it will be omitted.

For the purposes of this section, the boundary conditions are said to be conservative if neither boundary is absorbing and at least one boundary is not purely adhesive, that is,

$$\kappa_0 = \kappa_1 = 0 , \qquad \pi_0 + \delta_0 + \pi_1 + \delta_1 > 0$$

Let $C(a)$ denote the mean cost per unit time of such a process corresponding to the admissible control $a \in M$. Then $C(a)$ is the unique number to which there exists a solution to (3), (4). The minimal mean cost $\hat{C}$ is defined to be

$$\hat{\Theta} = \inf_{a \in M} \Theta(a) .$$

An admissible control $a \in M$ is said to be an optimal control if $\Theta(a) = \hat{\Theta}$. The main result for conservative processes and undiscounted costs is the following.

<u>Theorem 8</u>. Suppose the boundary conditions are conservative. The minimal mean cost is the unique number $\hat{\Theta}$ such that the equation

$$w'(s) + \min \{d(s,a)^{-1}[b(s,a)w(s) - \hat{C} + c(s,a)]\} = 0 \tag{15}$$

has a solution $w(\cdot)$ satisfying

$$\theta_j \int_S \left[\int_{r_j}^{s} w(y)dy + \nu_j(s) \right] du_j(s) + (-1)^j \pi_j w(r_j) + \sigma_j(\gamma_j - \hat{\Theta}) = 0 , \qquad j = 0,1, \tag{16}$$

where $\gamma_j = \min_{a \in A_{r_j}} c(r_j,a), \quad j = 0,1$.

A control $a \in M$ is optimal if and only if

$$\begin{aligned} &d(s,a(s))^{-1}[b(s,a(s))w(s) - \hat{\Theta} + c(s,a(s))] \\ &= \min_{a \in A_s} \{d(s,a)^{-1}[b(s,a)w(s) - \hat{\Theta} + c(s,a)]\} \end{aligned} \tag{17}$$

for every $s \in S$ which is a continuity point of $a(\cdot)$ and

(18) $$\sigma_j(c(r_j, a(r_j)) - \gamma_j) = 0 , \qquad j = 0,1 .$$

Proof. Equation (15) has a solution satisfying (16) for a unique number Θ by Theorem 16 of Chapter III. If $\Theta(a) < \Theta$ then a contradiction can be obtained as was done with Theorem 3 of Chapter III to show the unicity of $v(r_0)$. Using the reasoning of Theorem 3, there exists a sequence $\{a_n(s)\}$ of admissible controls such that $\Theta(a_n) \to \Theta$ as $n \to \infty$, so $\Theta = \inf_{a \in M} \Theta(a)$.

To prove the necessary and sufficient condition for a control to be optimal, if (17) and (18) are true, the $w(s)$ and Θ satisfy (3), (4) so $\Theta = \Theta(a)$ and $a(s)$ is optimal. Conversely, if $a(s)$ is optimal but (17) is violated at a continuity point of $a(s)$, then employing the reasoning of Theorem 3 of Chapter III used to show the unicity of $v(r_0)$, we construct a process with negative continuous movement costs, non-positive adhesion costs, and zero instantaneous return costs but with a zero mean cost per unit time, a contradiction. Finally, if $\Theta = \Theta(a)$ and (17) holds but (18) is violated, then a similar contradiction is obtained, and Theorem 8 is proved.

4. Existence of Admissible Optimal Controls

There is no guarantee that an admissible optimal control will exist for a controlled diffusion process. A piecewise continuous optimal control need not exist, as the following example shows: $S = A_s = [-1,1]$

d(s,a) and b(s,a) constants, and $c(s,a) = sa \sin(\frac{1}{s})$. Conditions which guarantee the existence of admissible optimal controls are provided in this section.

It is apparent that the problem of controlling a diffusion process has several similarities to the problem of controlling a classical deterministic system. If the expected discounted cost v(s) of a diffusion process is interpreted as the state as a function of time of a deterministic system, then differential equation (1) describes the behavior of this system over the time interval $[r_0, r_1]$ subject to the (nonclassical) boundary condition (2). Furthermore, the problem of optimally controlling this deterministic system so as to minimize the functional

$$\int_{r_0}^{r_1} v(s)\,ds$$

is equivalent to the problem of optimally controlling the diffusion process. Thus the question of whether piecewise continuous optimal controls exist for deterministic systems is germane to the subject of this section.

The mathematical system theory literature pertaining to the existence of optimal controls can generally be classified into one of two categories. Most research (e.g., Cesari [4]) has been concerned with proving the existence of a measurable optimal control. These papers do not concern us because measurable controls need not be admissible from our standpoint. On the other hand, a few papers (e.g., Halkin [11]) prove the existence of piecewise continuous optimal controls, but always for the special case where the state equation is linear in the state and control together. The following results will serve to weaken this linearity restriction, although it should be borne in mind that the classical results are for N-dimensional systems. The existence theorems in this section will all be for the discounted cost case; analogous results hold for the undiscounted cost case.

We say the function $f(s) : S \to E$ is analytic at $\bar{s}$ if it has an absolutely convergent power series expansion $f(s) = \sum_{j=0}^{\infty} a_j s^j$ in some neighborhood of $\bar{s}$. The function $f(s)$ is analytic on the interval $S_1 \subset S$ if there exists an open interval $S_2 \supset S_1$ and a function $g(s)$ which is analytic at each $s \in S_2$ such that $f(s) = g(s)$ for each $s \in S_1$. The function $f(s)$ is piecewise analytic on S if S can be decomposed into a finite number of intervals on each of which $f(s)$ is analytic. The following theorem is the main result of this section.

Theorem 9. If $A_s = \{1,2,\cdots,N\}$ and if $d(\cdot,a)$, $b(\cdot,a)$ and $c(\cdot,a)$ are piecewise analytic on S for each fixed $a = 1,2,\cdots,N$, then there exists a piecewise constant optimal control.

Proof. The function $d(s,a)$ is positive for all $s \in S$ and all $a = 1,\cdots,N$ so $\alpha(s,a) = d(s,a)^{-1}$, $\beta(s,a) = b(s,a)d(s,a)^{-1}$, and $\gamma(s,a) = c(s,a)d(s,a)^{-1}$ are piecewise analytic in s on S for each fixed $a = 1,\cdots,N$. Let $v(s)$ be the minimal expected discounted cost for this process. Let $a(s)$ be a function on S which satisfies $a(s) \in \{1,\cdots,N\}$ and

$$v''(s) = -\beta(s,a(s))v'(s) + \lambda\alpha(s,a(s))v(s) - \gamma(s,a(s))$$

for each $s \in S$. Thus $a(s)$ will be an admissible optimal control if it can be chosen piecewise constant. To prove this choice is possible, it suffices to show for each $\bar{s} \in S$ the existence of some $\delta > 0$ such that $a(s)$ can be chosen constant on $(\bar{s} - \delta, \bar{s}) \cap S$ and on $(\bar{s}, \bar{s} + \delta) \cap S$. We discuss only the second case, leaving the other to the reader.

For $i = 1, \cdots, N$ and arbitrary $\bar{s} \in [r_0, r_1)$, let $v_i(s)$ be the unique solution on S to

$$v_i''(s) = -\beta(s,i)v_i'(s) + \lambda\alpha(s,i)v_i(s) + \gamma(s,i)$$

and

$$v_i(\bar{s}) = v(\bar{s}), \quad v_i'(\bar{s}) = v'(\bar{s}).$$

From differential equation theory, $v_i(s)$ is piecewise analytic and therefore analytic on $(\bar{s}, \bar{s} + \delta)$ for some $\delta > 0$. Hence for some $\delta > 0$ there exists some integer j in $\{1, \cdots, N\}$ such that $v_j(s) \geq v_i(s)$ for all $i = 1, \cdots, N$ and all $s \in (\bar{s}, \bar{s} + \delta)$. We shall now show that the action $a(s) = j$ is optimal for all small enough $s > \bar{s}$.

For this j and each $i = 1, \cdots, N$ define

$$\psi_i(s) = [\beta(s,i) - \beta(s,j)]v_j'(s) - \lambda[\alpha(s,i) - \alpha(s,j)]\, v_j(s) + [\gamma(s,i) - \gamma(s,j)].$$

and note that $\psi_i(s)$ is analytic on $(\bar{s}, \bar{s} + \delta)$ for small enough $\delta > 0$. If we denote $\tilde{v}_i(\bar{s}) = v_j(s) - v_i(s)$, then $\tilde{v}_i(s)$ will be the unique solution on S to

$$\tilde{v}_i''(s) = -\beta(s,i)\tilde{v}_i'(s) + \lambda\alpha(s,i)\tilde{v}_i(s) + \psi_i(s) ,$$

satisfying $\tilde{v}_i(\bar{s}) = \tilde{v}_i'(\bar{s}) = 0$. For some $\delta > 0$, $\psi_i(s)$ is either uniformly positive, uniformly negative, or vanishes identically on $(\bar{s}, \bar{s} + \delta)$. By differential equation theory and the choice of j, we conclude for some $\delta > 0$ and each $i = 1, \cdots, N$ that $\psi_i(s) \geq 0$ for all $s \in (\bar{s}, \bar{s} + \delta)$.

It follows that for all small enough $s > \bar{s}$ we have

$$v_j''(s) = - \min_{i \in \{1, \cdots, N\}} [\beta(s,i)v_j'(s) - \lambda\alpha(s,i)v_j(s) + \gamma(s,i)] ,$$

$$v_j(\bar{s}) = v(\bar{s}) , \quad \text{and} \quad v'(\bar{s}) = v'(\bar{s}) .$$

Because of the uniqueness of solutions to this equation, this implies $v(s) = v_j(s)$ for all small enough $s > \bar{s}$. Hence we can choose $a(s) = j$ as the optimal action for each small enough $s > \bar{s}$.

Corollary 10 Let $A_s = \{f_1(s), \cdots, f_N(s)\}$ and suppose for $i = 1, \cdots, N$ that $f_i(s)$ is a bounded, piecewise continuous, vector-valued function on S and the functions $a(s, f_i(s))$, $b(s, f_i(s))$, and $c(s, f_i(s))$ are piecewise analytic on S. Then a piecewise continuous, admissible optimal control exists.

Proof. If we set $c(s,i) = c(s,f_i(s))$, etc., so that $\tilde{A}_s = \{1,\cdots,N\}$, then the proof is immediate.

An admissible optimal control will therefore exist if, among other things, the map A_s is finite valued and piecewise continuous. Note that the number of available actions may vary from state to state. The hypotheses of Corollary 10 are satisfied by a variety of functions. For example, if $d(s,a)$ is an analytic function of the two variables s and a, and $f_i(s)$ is analytic in s, then $d(s,f_i(s))$ is analytic because the composition of analytic functions is analytic. On the other hand, suppose $d(s,a)$ is an analytic function of s for each fixed a, but it is not analytic in the two variables s and a together. If $f_i(s)$ is a piecewise constant function, then $d(s,f_i(s))$ is piecewise analytic.

The following theorem exploits the fact that if the optimal action is unique for all but a finite number of $s \in S$, then a piecewise continuous optimal control must exist.

Theorem 11. Let $v(s)$ be the minimal expected discounted cost, suppose A_s is a convex set for all but a finite number of $s \in S$, and suppose $d(s,a)^{-1}[b(s,a)v'(s) - \lambda v(s) + c(s,a)]$ is a strictly convex function of a for all but a finite number of $s \in S$. Then an admissible optimal control exists.

Proof. Let the unique (apart from a finite number of points) control $a(\ ) : S \to K$ be such that $a(s) \in A_s$ and

$$d(s,a(s))^{-1}[b(s,a(s))v'(s) - \lambda v(s) + c(s,a(s))]$$

$$= \min_{a \in A_s} \{d(s,a)^{-1}[b(s,a)v'(s) - \lambda v(s) + c(s,a)]\}$$

for all $s \in S$. In view of the uniqueness, $a(\cdot)$ is piecewise continuous, completing the proof.

The proof of Corollary 12 is an immediate consequence of Theorem 11 and is therefore omitted.

Corollary 12. If A_s is a convex set for all but a finite number of $s \in S$, $c(s,a)d(s,a)^{-1}$ is strictly convex in a for all but a finite number of $s \in S$, and $d(s,a)^{-1}$ and $b(s,a)d(s,a)^{-1}$ are affine with respect to a, then an admissible optimal control exists.

The following theorem combines elements of the previous two.

Theorem 13. Let $A_s = [a_1(s), a_2(s)]$, a compact interval in E for each $s \in S$, where $a_1(s) \leq a_2(s)$ are bounded, piecewise continuous functions. Suppose $d(s,a_i(s))$, $b(s,a_i(s))$, and $c(s,a_i(s))$ are piecewise analytic on S for $i = 1,2$. Let S be decomposed into a finite number of intervals. If S_1 is any such interval, then suppose that one of the three functions $d(s,a)^{-1}$, $b(s,a)d(s,a)^{-1}$, or $c(s,a)d(s,a)^{-1}$ is either strictly convex or strictly concave in a, and the other two functions are affine in a, for all $s \in S_1^0$. Then an admissible optimal control exists.

Proof Define $\alpha(s,a) = d(s,a)^{-1}$, $\beta(s,a) = b(s,a)d(s,a)^{-1}$, and $\gamma(s,a) = c(s,a)d(s,a)^{-1}$. We can assume without loss of generality that $a_1(s)$ and $a_2(s)$ are continuous on S; that $\alpha(s,a_i(s))$, $\beta(s,a_i(s))$, and $\gamma(s,a_i(s))$ are analytic on S for $i = 1,2$; and that the decomposition is the trivial one, $S = \ddot{S}$. There will then be three cases, corresponding to which of the three functions $\alpha(s,a)$, $\beta(s,a)$ or $\gamma(s,a)$ is strictly concave or strictly convex. Throughout, we use the fact that $v''(s)$ is continuous on S (Lemma 5, Chapter III).

Case (i): $\gamma(s,a)$ strictly concave or strictly convex in a.

If $\gamma(s,a)$ is strictly concave in a, then so is $\beta(s,a)v'(s) - \lambda\alpha(s,a)v(s) + \gamma(s,a)$. This function is minimized by either the action $a = a_1(s)$ or $a = a_2(s)$, so this case reduces to the finite action situation and Corollary 10 applies. The situation where $\gamma(s,a)$ is strictly convex is covered by Corollary 12.

Case (ii): $\alpha(s,a)$ strictly concave or strictly convex in a.

By reasoning similar to the above, an admissible control is optimal provided $v(s)$ changes from zero to a non-zero value only a finite number of times as s increases from r_0 to r_1. Suppose not, so that in any neighborhood of $\bar{s} \in S$, say, the function $v(s)$ changes sign infinitely often as $s \to \bar{s}$. By continuity, it is necessary that $v(\bar{s}) = v'(\bar{s}) = \min_{a \in A_{\bar{s}}} \gamma(\bar{s},a) = 0$. Since $\gamma(s,a)$ is affine in a,

$\min_{a \in A_s} \gamma(s,a) = \min_{i=1,2} \gamma(s, a_i(s))$. By analyticity, $\min_{a \in A_s} \gamma(s,a)$ is either zero, positive or negative for all small enough $s > \bar{s}$, we examine these three situations. In the first situation we must have for all such $s > \bar{s}$ that $v(s) = 0$, a contradiction. Secondly, if $\min_{a \in A_s} \gamma(s,a) > 0$ for all small enough $s > \bar{s}$, there will exist a sequence $s_j \downarrow \bar{s}$ such that $v(s_j) \leq 0$, $v'(s_j) = 0$, and $v''(s_j) \geq 0$. But for any large enough j and some $a_j \in A_{s_j}$, we have

$$v''(s_j) = -\beta(s_j,a_j)v'(s_j) + \lambda\alpha(s_j,a_j)v(s_j) - \gamma(s_j,a_j)$$

$$\leq -\gamma(s_j,a_j) \leq -\min_{a \in A_{s_j}} \gamma(s_j,a) < 0$$

by assumption, a contradiction. As the third and final situation, suppose $\min_{a \in A_s} \gamma(s,a) < 0$ for all small enough $s > \bar{s}$. There exists a sequence $s_j \downarrow \bar{s}$ such that $v(s_j) \geq 0$, $v'(s_j) = 0$, and $v''(s_j) \leq 0$. There also exists a corresponding sequence $a_j \in A_{s_j}$ such that for all large enough j, $\gamma(s_j,a_j) < 0$. By the optimality condition for any s_j

$$-v''(s_j) \leq -\lambda\alpha(s_j,a_j)v(s_j) + \gamma(s_j,a_j) \leq \gamma(s_j,a_j)$$

Thus for all large enough j we have $v''(s_j) > 0$, a contradiction.

Case (iii). $\beta(s,a)$ strictly concave or strictly convex in a

By reasoning similar to the above, an admissible control is optimal if $v'(s)$ changes from zero to a non-zero value only a finite number of times as s increases from r_0 to r_1. Suppose not, so that in any neighborhood of $\bar{s} \varepsilon S$, say, the function $v'(s)$ changes sign infinitely often as $s \downarrow \bar{s}$. By continuity, we must have $v'(\bar{s}) = v''(\bar{s}) = 0$ Define $y(s) = v(s) - v(\bar{s})$ so that $y(\bar{s}) = y'(\bar{s}) = 0$ and

$$y''(s) = -\min_{a \varepsilon A_s} \{\beta(s,a)y'(s) - \lambda\alpha(s,a)y(s) + \bar{\gamma}(s,a) ,$$

where $\bar{\gamma}(s,a) = \gamma(s,a) - \lambda\alpha(s,a)v(s)$. Note that $\bar{\gamma}(s,a)$ satisfies the same hypotheses as $\gamma(s,a)$. In particular, $\min_{a\varepsilon A_s} \bar{\gamma}(s,a) = \min_{i=1,2} \bar{\gamma}(s,f_i(s))$ and this function is either positive, zero, or negative for all small enough $s > \bar{s}$. We now examine a hierarchy of situations.

If $\min_{i=1,2} \bar{\gamma}(s,f_i(s)) > 0$ for all small enough $s > \bar{s}$, then by differential equation theory we have that $y(s) > 0$ and $y'(s) \geq 0$ for all such s. Thus there exists some $s_0 > \bar{s}$ in this neighborhood with $y'(s_0) = 0$ and $y(s_0) > 0$. Then by Lemma 10 of Chapter III we have $y'(s) > 0$ for all $s > s_0$ in the specified neighborhood of $\bar{s}$, a contradiction. If $\bar{\gamma}(s,f_1(s))$ and $\bar{\gamma}(s,f_2(s))$ are both non-positive for all small enough $s > s_0$, then a similar contradiction is obtained.

As the final situation we need to consider, suppose $\bar{\gamma}(s,f_1(s)) > 0$ but $\bar{\gamma}(s,f_2(s)) < 0$ for all small enough $s > \bar{s}$ (the proof for $\bar{\gamma}(s,f_1(s)) < 0 < \bar{\gamma}(s,f_2(s))$ is similar and left to the reader). Define $w(s) = \bar{\gamma}(s,f_2(s))/[\lambda\alpha(s,f_2(s))]$ Now $w(s) < 0$ for all small enough $s > \bar{s}$, and $v''(\bar{s}) = 0$ implies $w(\bar{s}) = 0$. Since $w(s)$ is analytic, we must have $w'(s) < 0$ for all small enough $s > \bar{s}$ First we'll show that $y(s) < 0$ for all small enough $s > \bar{s}$ If $s > \bar{s}$ is such that

$y(s) \geq 0$ and $y'(s) = 0$ then

$$\begin{aligned} y''(s) &\geq \lambda\alpha(s,f_2(s))y(s) - \bar{\gamma}(s,f_2(s)) \\ &> \lambda\alpha(s,f_2(s))w(s) - \bar{\gamma}(s,f_2(s)) = 0 . \end{aligned}$$

The continuity of $y''(s)$ and $y(s) \geq 0$ imply $y'(s) > 0$ for all small enough $s > \bar{s}$, a contradiction.

We now assume $y(s) < 0$ for all small enough $s > \bar{s}$ and show $y'(s) = 0$ implies $y''(s)$ has the same sign as $y(s) - w(s)$. This fact is immediate if $f_2(s)$ is an optimal action at this s. Alternatively, if $f_1(s)$ but not $f_2(s)$ is optimal, then

$$\begin{aligned} -\lambda\alpha(s,f_2(s))w(s) + \bar{\gamma}(s,f_2(s)) &= 0 \\ < -\lambda\alpha(s,f_1(s))y(s) + \bar{\gamma}(s,f_1(s)) &= -y''(s) \\ < -\lambda\alpha(s,f_2(s))y(s) + \bar{\gamma}(s,f_2(s)) &, \end{aligned}$$

so $y''(s) < 0$ and $y(s) < w(s)$.

We now make the concluding arguments by considering the three situations corresponding to the sign of $y'(s)$ for all small enough $s > \bar{s}$. First, we have $y(s) < 0$, so $y'(s) \geq 0$ for all small enough $s > \bar{s}$ leads to a contradiction. Secondly, if $y'(s) \succeq 0$ for all small enough $s > \bar{s}$, then $y'(s) = 0$ implies $f_2(s)$ is the unique optimal action, because otherwise $f_1(s)$ is optimal, $y''(s) < 0$, and, by the continuity of $y''(s)$, a contradiction is obtained. Hence $y'(s) \leq 0$ for all small enough $s > \bar{s}$ implies some admissible control is optimal.

This is because if $\beta(s,a)$ is convex with respect to a then $\beta(s,a)y'(s) - \lambda\alpha(s,a)y'(s) + \bar{\gamma}(s,a)$ is concave and by earlier reasoning, the optimal control is piecewise continuous in this neighborhood. On the other hand, if $\beta(s,a)$ is strictly concave with respect to a then the optimal action is unique for each small enough $s > \bar{s}$, in which case the optimal control is continuous in this neighborhood.

As the final situation, suppose $y'(s)$ assumes both positive and negative values in every neighborhood of $\bar{s}$. By the preliminaries above, there exists a sequence $\{s_j\} \downarrow \bar{s}$ such that $y'(s_j) = 0$; j even implies $y''(s_j) \geq 0$, $y(s_j) \geq w(s_j)$, and $y'(s) \geq 0$ for $s \in [s_j, s_{j-1}]$; and j odd implies $y''(s_j) \leq 0$, $y(s_j) \leq w(s_j)$, and $y'(s) \leq 0$ for $s \in [s_j, s_{j-1}]$. Thus if j is even we have $w(s)$ crossing $y(s)$ from below as s increases from s_j to s_{j-1}. Since $y'(s) \geq 0$ for all such s we must have $w'(s) \geq 0$ for some such s, which is a contradiction for large enough even j. Hence in every situation either some admissible control is optimal or a contradiction can be obtained, and Theorem 13 is proved.

Corollary 14. Let the hypotheses of Theorem 13 be satisfied except that, if S is an interval with $d(s,a)^{-1}$ and $b(s,a)d(s,a)^{-1}$ affine, then $c(s,a)d(s,a)^{-1}$ is either affine, concave, or strictly convex in a for all $s \in S^0$. Then an admissible optimal control exists.

Proof. The proof of Case (i) in the proof of Theorem 13 goes through without change.

Corollary 15. Let the hypotheses of Theorem 13 be satisfied except that if $\tilde{S}$ is an interval in the decomposition of S then one of the three functions $d(s,a)^{-1}$, $b(s,a)d(s,a)^{-1}$, $c(s,a)d(s,a)^{-1}$ is analytic in (s,a) jointly on $\tilde{S} \times K$ and either concave or convex but not affine in a for all $s \in \tilde{S}^0$, and the other two functions are affine in a for all $s \in \tilde{S}^0$. Then an admissible optimal control exists.

Proof. In view of the proof of Theorem 13, it suffices to show that if $f(s,a)$ is some function which is analytic in (s,a) jointly on $\tilde{S} \times K$ and concave but not affine in a for all $s \in \tilde{S}^0$, then $f(s,a)$ is strictly concave in a for all but a finite number of $s \in \tilde{S}$ (the proof for $f(s,a)$ convex is similar and left to the reader). Since $f(s,a)$ is not affine in a, there exists some $\bar{s} \in \tilde{S}^0$ such that $f(\bar{s},a)$ is not affine. The analytic function $\frac{\partial^2}{\partial a^2}f(\bar{s},a)$ is thus negative for all but a finite number of $a \in K$ including, say, $\bar{a} \in A_{\bar{s}}$. It follows that the analytic function $\frac{\partial^2}{\partial a^2}f(s,\bar{a})$ is negative for all but a finite number of $s \in \tilde{S}$, in which case we must have $\frac{\partial^2}{\partial a^2}f(s,a)$ non-zero and $f(s,a)$ strictly concave in a for all but a finite number of $s \in \tilde{S}$.

5. Application: Control of a Dam

Suppose that the water level of a reservoir behaves like a stationary Markov process and fluctuates indefinitely in a continuous fashion between the numbers $r_0 < r_1$, which correspond respectively to the bottom of the reservoir and the top of the dam. Furthermore, suppose that the water level can be controlled to a certain extent by discharging water from the reservoir and that there exists a cost or utility rate associated with alternative discharge rates. Finally, suppose that there exists a second cost associated with the water level being at a particular value for one unit of time. Then the problem of optimally controlling this reservoir system may perhaps be stated as the problem of optimally controlling a conservative diffusion process.

It will be assumed that the water level of the reservoir behaves like a controlled diffusion process with reflection, or possibly reflection combined with adhesion, at each of the boundaries r_0 and r_1. The control action corresponds to the rate of water discharge through the dam and the control will be a piecewise continuous function of the water level. In addition, it is assumed that the costs of the reservoir system can be represented by a continuous movement cost, that is, the sum of the control and water level cost rates will be a continuous function of the discharge rate and the water level. Thus the diffusion process will be conservative, and in the case of undiscounted costs the optimal control will be that admissible control which yields the minimum expected cost per unit time. In the discounted cost case the optimal control will yield the minimum expected discounted cost, which will be a function of the initial water level.

This model is essentially a generalization of one by Bather [1]. His model assumes the reservoir input rate behaves like ordinary Brownian motion with positive drift, that a cost rate is associated with alternative discharge rates but not with alternative water levels, and that all controls must be continuous functions of the water level. In addition, if his water level process becomes negative then he assumes the water level is actually zero, so that pure reflection is impossible. Finally, his only optimality criterion is that of maximizing the expected utility per unit time.

<u>Example</u>. This example of controlling a reservoir uses the discounted cost criterion for evaluating optimality. The state space S and the set-valued function A_s of admissible actions equal the unit interval. The diffusion coefficient equals the positive constant A; and the drift coefficient equals $B(1 - 2a)$, where B is a positive constant. The continuous movement cost is a convex quadratic function of the water level, namely $ps^2 - ps + q$, where $p > 0$ and q are arbitrary numbers. The boundary conditions, at $s = 0$ and $s = 1$, are pure reflection. The intuitively obvious control is to try to maintain the water level at $s = \frac{1}{2}$, that is, maintain a minimum discharge rate $(a = 0)$ when the water level is less than $\frac{1}{2}$ and maintain a maximum discharge rate $(a = 1)$ otherwise. If this conjecture is correct then by symmetry the derivative of the minimal expected discounted cost will equal zero at $s = \frac{1}{2}$. Solving equation (1) on $[0, \frac{1}{2}]$ with $a(s) = v'(0) = v'(\frac{1}{2}) = 0$, we obtain the solution

$$v(s) = L_1 e^{\lambda_1 s} + L_2 e^{\lambda_2 s} + \frac{1}{\alpha}(ps^2 - ps + q)$$

$$+ \frac{2Ap}{\alpha^2} + \frac{B}{\alpha^2}(2ps - p) + \frac{2B^2 p}{\alpha^3}, \quad s \in [0, \tfrac{1}{2}]$$

where
$$L_1 = \frac{1}{\ell_1\left(e^{\frac{\ell_1}{2}} - e^{\frac{\ell_2}{2}}\right)}\left[-\frac{p}{\lambda}e^{\frac{\ell_2}{2}} + \frac{2pB}{\lambda^2}\left(e^{\frac{\ell_2}{2}} - 1\right)\right],$$

$$L_2 = \frac{1}{\ell_2\left(e^{\frac{\ell_1}{2}} - e^{\frac{\ell_2}{2}}\right)}\left[\frac{p}{\lambda}e^{\frac{\ell_1}{2}} + \frac{2pB}{\lambda^2}\left(1 - e^{\frac{\ell_1}{2}}\right)\right],$$

$$\ell_1 = \frac{-A^{-1}B + \sqrt{A^{-2}B^2 + 4A^{-1}\lambda}}{2},$$

and
$$\ell_2 = \frac{-A^{-1}B - \sqrt{A^{-2}B^2 + 4A^{-1}\lambda}}{2}.$$

Similarly, solving (1) on $[\frac{1}{2},1]$ with $a(s) = 1$ and $v'(\frac{1}{2}) = v'(1) = 0$ we obtain

$$v(s) = N_1 e^{-\ell_1 s} + N_2 e^{-\ell_2 s} + \frac{1}{\lambda}(ps^2 - ps + q)$$
$$+ \frac{2Ap}{\lambda^2} - \frac{B}{\lambda^2}(2ps - p) + \frac{2B^2 p}{\lambda^3} \qquad s \in [\tfrac{1}{2},1],$$

where
$$N_1 = \frac{e^{\frac{\ell_1}{2}}}{\ell_1\left(e^{-\frac{\ell_1}{2}} - e^{-\frac{\ell_2}{2}}\right)}\left[\frac{p}{\lambda} + \frac{2pB}{\lambda^2}\left(e^{-\frac{\ell_2}{2}} - 1\right)\right].$$

and
$$N_2 = \frac{e^{\frac{\ell_2}{2}}}{\ell_2\left(e^{-\frac{\ell_1}{2}} - e^{-\frac{\ell_2}{2}}\right)}\left[-\frac{p}{\lambda} + \frac{2pB}{\lambda^2}\left(1 - e^{-\frac{\ell_1}{2}}\right)\right].$$

After verifying that $v(s)$ is continuous at $s = \frac{1}{2}$, we conclude that $v(s)$ is the expected discounted cost corresponding to the admissible control $a(s) = 0$ for $0 \le s < \frac{1}{2}$ and $a(s) = 1$ for $\frac{1}{2} \le s < 1$. It remains to show that $a(s)$ is optimal. Since it can be shown that $v(s) = v(1 - s)$ for all $s \in [0,\frac{1}{2}]$, by (9) it suffices to show that $v'(s) \le 0$ for all $s \in [0,\frac{1}{2}]$. Since the continuous movement cost $ps^2 - ps + q < q$ for all $s \in (0,1)$, we must have $v(s) < \int_0^\infty e^{-\lambda t} q\,dt$ for all $s \in S$. In particular, $\lambda v(0) < q$. Similarly, $\lambda v(\frac{1}{2}) > q - \frac{1}{4}p$. It follows that $v''(0) < 0 < v''(\frac{1}{2})$. We know that $\ell_1 > 0$, $\ell_2 < 0$, and $L_1 < 0$. If $L_2 \ge 0$, then $v'(s)$ would be concave, a contradiction. Thus, with $L_2 < 0$, there exists some $\bar{s} \in [0,\frac{1}{2}]$ such that $v'(s)$ is convex for $0 \le s \le \bar{s}$ and is concave for $\bar{s} \le s \le \frac{1}{2}$. If $v'(s) > 0$ for any $s \in [0,\frac{1}{2}]$, then a contradiction is obtained, so we must have $a(s)$ optimal and $v(s)$ the minimal expected discounted cost.

6 Application: Control of Pollution.

Suppose that the index of pollution is constrained to fall between zero and some positive number. This would be the case, for example, when dealing with an air basin or a body of water. Assume that a factory, a collection of automobiles, or a similar polluting mechanism wants to control this index of pollution by optimally choosing the amount of its waste products that is being emitted as a pollutant as opposed to being processed in a pollution free manner. Finally, assume that there exists a cost to the controller for each level of control as well as to each value of the pollution index. Then the class of controlled diffusion processes used as models of dam-reservoir systems may perhaps be used as models of pollution systems.

The state of the process will correspond to the index of pollution, and the boundary behavior, at zero and the maximum index value, will be reflection or possibly reflection combined with adhesion. An admissible control will be a piecewise continuous function of the state space that will represent the portion of the controller's wastes that is being emitted as a pollutant. Presumably, (i) the bigger the control value the smaller the control cost rate (less needs to be processed), (ii) the bigger the control value the bigger the drift coefficient; and (iii) the bigger the pollution index the greater the pollutant cost rate. An optimal control will be an admissible control which yields either the minimal expected discounted cost or the minimal mean cost per unit time.

The choice of the proper upper boundary condition is open to question. One possibility other than reflection is absorption, with the interpretation that in the rare event the pollution ever reaches a sufficiently high, intolerable level, then a "disaster" would occur at some high cost. If it can be assumed that the pollution index rarely, if ever, attains its upper limit then the choice of the upper boundary condition becomes moot. This might be the case if the pollution index is thought of as the percent of the natural medium which has been replaced by pollutants. The oxygen in an air basin would never be completely replaced by smog, for example, so the drift and diffusion coefficients of the corresponding diffusion process model would be chosen accordingly. Then the process would rarely be affected by the upper boundary, so its effects could largely be ignored.

Example. This example of controlling pollution uses the discounted cost criterion for evaluating optimality. The state space S and the set-valued function A_s of admissible actions equal the unit interval. The diffusion coefficient, drift coefficient, and continuous movement cost equal respectively A, $B(2a - 1)$, and $Cs + D(1 - a)$, where A, B, C, and D are arbitrary positive constants. Assume the boundary conditions are equivalent to pure reflection. In view of the boundary conditions, the solution of (5) for the minimal expected discounted cost $v(s)$ must be such that, in some neighborhoods of the boundaries, $v'(s) < D/2B$ and the optimal control $a(s)$ equals one. If $a(s) = 1$ for all $s \in S$ then

$$v(s) = L_1 e^{\ell_1 s} + L_2 e^{\ell_2 s} + \frac{Cs}{\lambda} + \frac{BC}{\lambda^2}$$

where

$$L_1 = \frac{C}{\lambda \ell_2}\left(\frac{1 - e^{\ell_2}}{e^{\ell_2} - e^{\ell_1}}\right);$$

$$L_2 = \frac{C}{\lambda \ell_2}\left(\frac{e^{\ell_1} - 1}{e^{\ell_2} - e^{\ell_1}}\right),$$

$$\ell_1 = \frac{-A^{-1}B + \sqrt{A^{-2}B^2 + 4A^{-1}\lambda}}{2},$$

and $$\ell_2 = \frac{-A^{-1}B - \sqrt{A^{-2}B^2 + 4A^{-1}\lambda}}{2} .$$

The derivative $v'(s)$ is then a concave function that equals zero at $s = 0$ and $s = 1$ and assumes its maximum value at

$$\bar{s} = \frac{1}{\ell_1 - \ell_2} \ln\left(\frac{-\ell_2^2 L_2}{\ell_1^2 L_1}\right) .$$

If $v'(\bar{s}) \leq \frac{D}{2B}$ then $v(s)$ equals the minimal expected discounted cost and the optimal control is to process none of the wastes at any level of pollution, that is, pollute as much as possible. Suppose, on the other hand, that $v'(\bar{s}) > D/2B$, that is,

$$\ell_1 L_1 e^{\ell_1 \bar{s}} + \ell_2 L_2 e^{\ell_2 \bar{s}} + \frac{C}{\lambda} > \frac{D}{2B} .$$

If $a(s) = 1$ for all $s \in [s_0, s_1]$, where $0 < s_0 < s_1 < 1$ and $v'(s_0) = v'(s_1) = D/2B$, then the derivative $v'(s)$ is concave on $[s_0, s_1]$, in which case $a(s) = 1$ is not optimal. Hence if $v'(\bar{s}) > 1$ there exist two numbers $0 < s_0 < s_1 < 1$ such that if $s \in [0, s_0] \cup (s_1, 1]$ then the optimal control is to pollute as much as possible, while if $s \in (s_0, s_1)$ then the optimal control is to process all of the waste products.

7. Application: Control of a Queueing System.

Suppose a queueing system is characterized by having a finite waiting room, that is, the length of the queue is less than or equal to some number. In addition, assume that the length of the waiting line can be controlled by the servers, such as by changing the service rate. Then let the number of customers in this queueing system be represented by a controlled diffusion process fluctuating between zero and the capacity of the queueing system. The behavior of the process at these boundaries can be either reflection or reflection combined with adhesion. The control action corresponds to the service mode, for example, the service rate. If there are costs associated with alternative queue lengths and control actions, then a control which yields the minimal expected cost of the diffusion model will be an optimal control of the queueing system. Presumably, the queue length cost will be an increasing function of the queue length, and controls with a greater tendency to shorten the queue length will be more expensive.

An obvious shortcoming of this model is the fact that the length of a queue is a discrete state process whereas the diffusion process is continuous. In cases where this continuous state approximation is not sufficiently accurate, however, it may be possible to construct a diffusion process so that a discrete process which can be extracted from it will have certain desired properties. This discrete process can be defined as follows. For some positive integer N, let the diffusion process be defined on $[0,N]$ and let the discrete process have $N+1$ states corresponding respectively to the integers $0,1,\cdots,N$. Then the discrete process will occupy state i if i was the most recent integer value attained by the continuous process sample path. The discrete process will then enter state $i + 1$ (or $i - 1$) at the epoch when the continuous process first attains the value $i + 1$ $(i - 1)$.

There is no assurance that the diffusion process can be constructed so that the first passage times of the extracted discrete process will have some specific probability distribution. In particular, exponentially distributed first passage times cannot generally be obtained. However, the following discussion will show that an arbitrary set of mean first passage times can be represented by the discrete process extracted from a Brownian motion with properly chosen boundary conditions.

Suppose that the queueing system has capacity N and that four items of data are specified: the transition probability p from state i to $i + 1$ and the mean occupation time δ in state i, for $i = 1,2,\ldots,N-1$; the mean first passage time t from state 0 to 1; and the mean first passage time u from state N to $N - 1$. We want to calculate the diffusion coefficient d, the drift coefficient b, and the boundary conditions $\pi_0, \sigma_0, \pi_1, \sigma_1$ (we let $\theta_0 = \theta_1 = \kappa_0 = \kappa_1 = 0$) so that the extracted discrete process will correspond to this queueing system in the specified manner.

Utilizing the fact that the expected state of the process upon exit from the interval $(i - 1, i + 1)$ given initial state i equals the product of the drift coefficient and the first passage time δ, we conclude that $b = (2p - 1)/\delta$. Utilizing standard diffusion process theory (see, for example, Mandl [16, pp. 100-102]) to calculate the mean first passage time in terms of the drift and diffusion coefficients, we conclude for $b \neq 0$ that the diffusion coefficient d must be the unique solution to $b = \coth(b/d) - \operatorname{csch}(b/d)$. Note that a unique solution for d always exists because $\coth(x) - \operatorname{csch}(x)$ increases monotonically from -1 to 1 as x increases from $-\infty$ to ∞. If we assume that $\delta < t$, $\pi_0 = 1$, and $b \neq 0$, then by standard diffusion process theory again the mean first passage time from state 0 to state 1 equals $\frac{d}{b}(\sigma_0 - \frac{1}{b})(1 - e^{-b/d}) + \frac{1}{b}$. Setting this equal to t allows one to calculate the coefficient σ_0 describing adhesion at boundary r_0. Similarly, if $b = 0$ then $d = 1/2\delta$ and $\sigma_0 = t - \delta$. The calculation of π_1 and σ_1 proceeds similarly.

Example. This example of controlling a queue uses the discounted cost criterion for evaluating optimality. We have $S = [0,N]$ and $A_s = [0,R]$. The diffusion coefficient equals the positive constant A, the drift coefficient equals $-a$, and the continuous movement cost equals $Cs + Da$, where C and D are positive constants. Thus larger control values will tend to shorten the queue length at the expense of a greater control cost. The boundary conditions are equivalent to pure reflection. The calculations for this example proceed similarly to those for the example in Section 6. In some neighborhoods of the boundaries the optimal control $a(s)$ must be zero. If $a(s) = 0$ for all $s \in S$ then the expected discounted cost is

$$v(s) = \frac{C}{\rho\lambda}\left(\frac{e^{-\lambda N} - 1}{e^{\lambda N} - e^{-\lambda N}}\right)e^{\lambda s} + \frac{C}{\rho\lambda}\left(\frac{e^{\lambda N} - 1}{e^{\lambda N} - e^{-\lambda N}}\right)e^{-\lambda s} + \frac{Cs}{\rho} \quad ,$$

where $\ell = \sqrt{A^{-1}\lambda}$. The derivative $v'(s)$ attains its maximum value at $s = N/2$, so if $v'(N/2) = \frac{-2C}{\lambda}\left(\frac{e^{\frac{\ell N}{2}} - e^{\frac{-\ell N}{2}}}{e^{\ell N} - e^{-\ell N}}\right) + \frac{C}{\lambda} \leq D$, then $a(s) = 0$ is optimal and $v(s)$ is the minimal expected discounted cost. On the other hand, if $v'(N/2) > D$, then there exist $0 < s_0 < s_1 < N$ such that if $s \in [0,s_0) \cup (s_1,N]$ then $a(s) = 0$ is optimal, whereas if $s \in (s_0,s_1)$ then $a(s) = R$ is optimal.

8 Application: Making Optimal Investments

Suppose the owner of an investment fund has available to him a number of alternative investment opportunities, each of which is characterized by a rate of return and a value of risk that are constant with respect to time. Moreover, suppose that the value of the investment fund is characterized by being bounded by two numbers. For example, the value might always be non-negative and if the fund's owner ever acquires a million dollars, then he would stop investing. If the owner wants to make the optimal choice of investments for every level of the fund's value, then his problem can perhaps be solved by the consideration of an appropriate controlled diffusion process.

Let the value of the investment fund correspond to the state of the diffusion process and assume that the behavior of the fund at the boundaries can be represented by some choice of diffusion process boundary conditions. For example, the fund value could behave like reflection at the lower boundary and absorption at the upper one. The map A_s describing admissible actions is formulated so that, for each $s \in S$, there is a one-to-one correspondence between admissible actions $a \in A_s$ and the investment opportunities which are available when the fund's value is s. Let the continuous movement cost reflect the utility to the fund's owner of the fund being at a particular level for one unit of time, and let the costs associated with the boundary conditions be defined in a corresponding manner. Then the control which yields the minimal expected cost for the controlled diffusion process will correspond to the optimal investment policy.

The value of risk associated with an investment is generally specified by the variance per dollar invested. Thus it is convenient to describe each investment opportunity by a pair (a_1,a_2) where $a_1 > 0$ is the variance per dollar invested and $a_2 \in E$ is the yield, that is, rate of return. Then we can let A_s be a compact subset of $E^+ \times E$ so that each admissible action $(a_1,a_2) \in A_s$ corresponds to some investment opportunity. Normally, the map A_s is a constant with respect to $s \in S$, but not necessarily so. Certain investment opportunities, for example, might be available only to funds of some minimum size.

In formulating this controlled diffusion process investment model, it remains to specify the drift and diffusion coefficients. Given a specific investment opportunity, the expected profit and standard deviation per unit time for a fund will be proportional to the fund's value. Consequently, if the fund is invested in opportunity $(a_1,a_2) \in A_s$ then the appropriate coefficients for the diffusion process model are $d(s,a) = s^2 a_1$ and $b(s,a) = sa_2$.

Example. For this optimal investment example we assume the fund's value is bounded as $0 < r_0 \leq s \leq r_1 < \infty$, and we want to minimize the probability of reaching r_0 before r_1. This problem can be solved by considering a non-conservative diffusion process with undiscounted costs.

Suppose each boundary condition is pure absorption with costs $\lambda_0 = 1$ and $\lambda_1 = 0$. Then with a zero continuous movement cost, the minimal expected cost $v(s)$ is the minimum probability of reaching r_0 before r_1 when starting at s. Clearly $v'(s) \leq 0$, so by (14), we want to choose $a \in A_s$ so that

$$\frac{b(s,a)}{d(s,a)} = \max_{a \in A_s} \frac{b(s,a)}{d(s,a)}$$

In particular, suppose A_s, $b(s,a)$ and $d(s,a)$ are as formulated above with A_s constant with respect to $s \in S$. If $a_2 > 0$ for some $(a_1, a_2) \in A_s$, then $\max_{a \in A_s} \left\{\frac{sa_2}{s^2 a_1}\right\} > 0$ and this corresponds to a favorable game. Note that if several investments have the same positive yield then the least risky one will maximize (sa_2/s^2a_1), so conservative play is optimal. On the other hand, suppose $a_2 \leq 0$ for all $(a_1, a_2) \in A_s$; this corresponds to an unfavorable game, e.g., a casino. In this case, if several investments have the same negative yield, then the riskiest one maximizes (sa_2/s^2a_1), that is, bold play is optimal.

CHAPTER III

MULTI-PERSON CONTROLLED DIFFUSIONS

This chapter generalizes the concept of a controlled one-dimensional diffusion process by allowing the process to be controlled by N persons. If the process is controlled by two persons with opposite objectives, then the problem of optimally controlling this process may be viewed as a zero sum, two-person game. On the other hand, if the process is controlled by $N \geq 2$ persons with possibly different objectives, then the problem of optimally controlling this process may be viewed as a non-zero sum, N-person game.

The results in this chapter are intimately connected with those for single person controlled diffusions (see Chapter II and Mandl [16]). In addition, minimax problems in the theory of diffusions have been treated by Girsanov [10]. The multi-person controlled diffusion process is formulated in the following section, the zero sum, two-person game problem is discussed in the succeeding four sections, and the non-zero sum, N-person game problem is treated in the final five sections. Both discounted and undiscounted costs are considered for both game problems, and existence theorems are provided. In addition, several possible applications of multi-person controlled diffusions are given. A major result of this chapter is that the value of a zero sum, two-person game is the unique solution of a differential equation

1. The Multi-Person Controlled Diffusion Process

The multi-person controlled diffusion process is formulated as in Chapter II, only taking into account the multiple number of controllers. Consider a diffusion process with state space S, a compact interval $[r_0, r_1]$ of the real line E, which is controlled by N persons (integer $N \geq 2$). For each $i = 1, 2, \cdots, N$, some positive integer n_i, and some compact set $K_i \subset E^{n_i}$, the i^{th} person's control is a vector-valued function on S with range K_i. Let A_s^i be a point-to-set map from S into K_i such that A_s^i is piecewise continuous in s in the Hausdorff metric and for each $s \in S$ the set A_s^i is a non-empty compact subset of K_i. Each time the process is observed in state s the i^{th} person chooses an action a_i from the set A_s^i. The set M_i of admissible controls for the i^{th} person consists of all piecewise continuous functions $a_i(s)$ on S with range in K_i such that the action $a_i(s) \in A_s^i$ for each $s \in S$.

Let $M = M_1 \times M_2 \times \cdots \times M_N$, $K = K_1 \times \cdots \times K_n$, $a(s) = (a_1(s), \cdots, a_N(s))$, and $A_s = (A_s^1, \cdots, A_s^N)$, so that M is the set of admissible controls, a function $a(s)$ is an admissible control if and only if $a(s) \in M$, and $a(\ ) \in M$ implies $a(s) \in A_s$ for each $s \in S$. Throughout this chapter it should be clear from the context whether the letter a denotes an admissible control $a = a(\ ) \in M$ or an admissible action $a \in A_s$ for some $s \in S$. The map A_s is characterized in Chapter II. We assume $M \neq \emptyset$ hereafter without further mention.

The definition of a multi-person controlled diffusion process is a slight generalization of Mandl's [16, p. 157] controlled diffusion process. Let $d(s,a)$ be a continuous, positive real-valued function on $S \times K$. Then for $a(\cdot) \in M$ the piecewise continuous function $d(s,a(s))$ is the diffusion coefficient of the process. Similarly, let $b(s,a)$ be a continuous real-valued function on $S \times K$ so that $b(s,a(s))$ is the drift coefficient of the diffusion process.

Following Mandl, with a given control $a(\cdot) \in M$ the multi-person controlled diffusion process is completely specified by the generalized classical differential operator

$$D \equiv d(s,a(s))\frac{d^2}{ds^2} + b(s,a(s))\frac{d}{ds}$$

together with Feller's [7,9] boundary condition

$$\kappa_j v(r_j) + \theta_j\left(v(r_j) - \int_S v(s)\,d\mu_j(s)\right) - (-1)^j \pi_j v'(r_j)$$

$$+ \sigma_j (Dv)(r_j) = 0, \qquad j = 0,1;$$

where $v(s)$ is some function whose second derivative is piecewise continuous on S. At each boundary r_0, r_1 the four non-negative parameters κ_j, σ_j, π_j and θ_j at least one of which must be positive correspond respectively to the phenomena of absorption, adhesion reflection and instantaneous return. Corresponding to θ_j is the probability distribution function $\mu_j(s)$ where $\int_{(r_0,r_1)} d\mu_j(s) = 1$. This boundary condition is interpreted more fully in Chapter II.

The multi-person controlled diffusion process generates costs according to its sample path and control (Mandl [16, p 148]). With the zero sum, two-person game problem, exactly one stream of costs is generated, as is the case with the single person controlled diffusion (Chapter II). But with the non-zero sum, N-person game problem, exactly N streams of costs will be generated, with one stream corresponding to each controller. The costs of a multi-person controlled diffusion will be formulated below for the N-person problem, but it should be borne in mind that the formulation for the zero sum problem is exactly the same, except that the subscript i relating the cost streams with the controllers will be dropped.

Each cost stream is comprised of the same three types of costs that were specified in Mandl [16] and Chapter II. The continuous movement cost for the i^{th} person is defined by the bounded, continuous real-valued function $c_i(s,a)$ on $S \times K_1 \times \cdots \times K_N$; let $c(s,a)$ denote the N-component vector of these functions. The cost for the i^{th} person due to instantaneous returns from boundary r_j is expressed by the real-valued function $\nu_{ji}(s)$ on S, which is integrable with respect to $\mu_j(s)$; let $\nu_j(s)$ denote the vector of these functions. Finally, the cost for the i^{th} person due to the termination (absorption) of the process at boundary r_j is λ_{ji}, and λ_j denotes the vector of these costs.

If $C_i(t)$ is the total of the i^{th} person's costs generated by the process up through time t, and $C(t) = (C_1(t), \ldots, C_N(t))$ is the vector of these costs, then the N-component vector

$$v(s) = E_s \int_0^{\infty} e^{-\lambda t} dC(t)$$

denotes the conditional expectation of the total discounted costs of the process given an initial state s, a control $a(\cdot) \in M$, and a discount factor $e^{-\lambda}$, $\lambda > 0$. From Mandl [16, p. 149] we have the following result.

<u>Theorem 1</u>. The vector of expected discounted costs corresponding to $a(\cdot) \in M$ is the unique function $v(s)$ on S such that $v'(s)$ is continuous,

$$(1) \qquad d(s,a(s))v''(s) + b(s,a(s))v'(s) - \lambda v(s) + c(s,a(s)) = 0$$

holds for every $s \in (r_0, r_1)$ which is a continuity point of $a(s)$, and

$$(2) \qquad (\theta_j + \kappa_j)v(r_j) - \theta_j \int_S (v(s) + \gamma_j(s))d\mu_j(s) - (-1)^j \pi_j v'(r_j)$$

$$+ \sigma_j(\lambda v(r_j) - c(r_j, a(r_j))) - \kappa_j \lambda_j = 0 , \qquad j = 0,1 .$$

If the process is non-conservative and neither boundary is purely adhesive, that is

$$\kappa_0 + \kappa_1 > 0 , \qquad \kappa_j + \pi_j + \theta_j > 0 , \qquad j = 0,1,$$

then by Mandl [16, p. 152] the vector $v(s) = E_s C(\infty)$ of the expected total undiscounted costs is finite and is the unique solution of (1) and (2) for $\lambda = 0$. If the process is conservative $(\kappa_0 + \kappa_1 = 0)$, then the total undiscounted costs may be infinite. The vector $\theta = (\theta_1, \ldots, \theta_N)$ in the following theorem, which is an immediate consequence of Mandl [16, pp. 152-157, 168], can be interpreted as the vector of mean costs per unit time.

Theorem 2. Let $\kappa_0 = \kappa_1 = 0$ and assume at least one boundary is not purely adhesive, that is, $\pi_0 + \theta_0 + \pi_1 + \theta_1 > 0$. If $v(s,\lambda)$ is the vector of expected discounted costs corresponding to $\lambda > 0$ and some $a(\cdot) \in M$, then

$$\lim_{\lambda \downarrow 0} \lambda v(s,\lambda) = \Theta \quad \text{and} \quad \lim_{\lambda \downarrow 0} \frac{d}{ds} v(s,\lambda) = w(s) ,$$

where Θ is some vector independent of the state s, and $w(s)$ is some absolutely continuous vector-valued function on S. Moreover,

$$P(\lim_{t \to \infty} t^{-1} C(t) = \Theta) = 1 ,$$

and (Θ, w) is the unique pair satisfying

$$(3) \qquad d(s,a(s))w'(s) + \rho(s,a(s))w(s) - \Theta + c(s,a(s)) = 0$$

for every $s \in (r_0, r_1)$ which is a continuity point of $a(s)$, and

$$(4) \qquad \theta_j \int_S \left\{ \int_{r_j}^{s} w(y)dy + v_j(s) \right\} d\mu_j(s) + (-1)^j \pi_j w(r_j)$$
$$+ \sigma_j (c(r_j, a(r_j)) - \Theta) = 0 , \qquad j = 0,1 .$$

2. The Zero Sum, Two-person Game Problem.

In this and the following three sections we consider diffusion processes which are controlled by two persons ($N = 2$), but which generate single streams of costs. The persons have opposite objectives; the first wants to minimize the costs while the other wants to maximize the costs. Note that this zero sum game can be regarded as a special case of the non-zero sum game problem for $N = 2$ by letting the second player's costs equal the negative of the first player's.

We shall consider a single stream of costs and therefore omit the subscript on all cost symbols. For any particular problem, player 1, who operates the first control component, endeavors to choose a control $a_1(\cdot) \in M_1$ so as to minimize the expected costs generated by the process. Player 2, who operates the second control component, endeavors to choose a control $a_2(\cdot) \in M_2$ so as to maximize the costs generated by the process. By a solution to this game is meant some admissible control which is a saddlepoint of the expected cost function. Thus, if player 1 unilaterally deviates from this optimal control, then the expected costs cannot be decreased but they may increase. Similarly, player 2 can unilaterally only decrease the expected costs.

The following two sections provide results respectively for the discounted cost case and the undiscounted cost case. The method for solving a problem is basically the same in each case. A differential equation is solved and the solution is used to determine the saddlepoint of a function with respect to all admissible controls. If this saddlepoint exists, then it is used to obtain an optimal control, that is, a solution to the zero sum, two-person game. Section 5 indicates a possible application of this model to optimal welfare policies.

3. The Zero Sum Problem with Discounted Costs.

Let $v(s,a_1,a_2) = v(s)$ denote the expected discounted cost of a process corresponding to the admissible control $a = (a_1,a_2) \in M$. Then $v(s,a_1,a_2)$ will be the unique solution of (1) and (2). The control $\tilde{a} \in M$ is said to be optimal if for all $a_1 \in M_1$, all $a_2 \in M_2$, and all $s \in S$ we have

$$v(s,\tilde{a}_1,a_2) \le v(s,\tilde{a}_1,\tilde{a}_2) \le v(s,a_1,\tilde{a}_2) ,$$

in which case $v(s,\tilde{a}_1,\tilde{a}_2)$ is said to be the value of the game. We shall later prove that the value of a game, if it exists, is provided by the following.

Theorem 3. There exists a unique solution $v(s)$ to

$$(5) \quad v''(s) + \min_{a_1 \in A_s^1} \max_{a_2 \in A_s^2} \{d(s,a_1,a_2)^{-1}[b(s,a_1,a_2)v'(s) - \lambda v(s) + c(s,a_1,a_2)]\} = 0$$

satisfying

$$(6) \qquad (\theta_j + \kappa_j)v(r_j) - \theta_j \int_S (v(s) + \nu_j(s))d\mu_j(s) - (-1)^j \pi_j v'(r_j)$$

$$+ \sigma_j(\lambda v(r_j) - \gamma_j) - \kappa_j \lambda_j = 0 , \qquad j = 0,1,$$

where $$\gamma_j = \min_{a_1 \epsilon A^1_{r_j}} \max_{a_2 \epsilon A^2_{r_j}} c(r_j,a_1,a_2) , \qquad j = 0,1 .$$

Before proving this theorem, some notation will be introduced and a number of preliminary lemmas will be proved.

Define: $$\alpha(s,a_1,a_2) = d(s,a_1,a_2)^{-1} ,$$

$$\beta(s,a_1,a_2) = b(s,a_1,a_2)d(s,a_1,a_2)^{-1} ,$$

$$\gamma(s,a_1,a_2) = c(s,a_1,a_2)d(s,a_1,a_2)^{-1} ,$$

$$g_1(s,v_1,v_2) = v_2 ,$$

$$g_1(s,v_1,v_2) = -\min_{a_1 \epsilon A^1_s} \max_{a_2 \epsilon A^2_s} \{\beta(s,a_1,a_2)v_2 - \lambda\alpha(s,a_1,a_2)v_1$$

$$+ \gamma(s,a_1,a_2)\} ,$$

and $$g(s,v_1,v_2) = \begin{pmatrix} g_1(s,v_1,v_2) \\ g_2(s,v_1,v_2) \end{pmatrix} .$$

We have the following result from Berge [2, pp. 115-116].

<u>Lemma 4</u>. Let X and Y be compact topological spaces. If f is a lower (upper) semi-continuous numerical function on $X \times Y$ and Γ is a

lower (upper) semi-continuous mapping of X into Y such that for each x, $\Gamma x \neq \phi$, then the numerical function h defined by

$$h(x) = \sup\{f(x,y) \mid y \in \Gamma x\}$$

is lower (upper) semi-continuous on X.

Lemma 5. If A_s^1 and A_s^2 are continuous at $\bar{s}$, and $(\bar{v}_1,\bar{v}_2)$ are arbitrary, then $g(\ ,v_1,v_2)$ is continuous in (s,v_1,v_2) at $(\bar{s},\bar{v}_1,\bar{v}_2)$.

Proof. For $i = 1,2$ let $C_i \subset E$ denote a compact set containing an open neighborhood of $\bar{v}_i$. With the notation of Lemma 4, identify X with $S \times K_1 \times C_1 \times C_2$, Y with K_2, f with $\beta(s,a_1,a_2)v_2 - \lambda\alpha(s,a_1,a_2)v_1 + \gamma(s,a_1,a_2)$, and Γ with A_s^2. Conclude by Lemma 4 that the numerical function

$$\max_{a_2 \in A_s^2} \{\beta(s,a_1,a_2)v_2 - \lambda\alpha(s,a_1,a_2)v_1 + \gamma(s,a_1,a_2)\}$$

is continuous at $(\bar{s},\bar{v}_1,\bar{v}_2)$ for all $a_1 \in K_1$. Repeating this reasoning in a similar manner, conclude that $g_2(s,v_1,v_2)$, and hence, trivially, $g(s,v_1,v_2)$, are continuous in (s,v_1,v_2) at $(\bar{s},\bar{v}_1,\bar{v}_2)$.

In the following lemma, we use the norm $\|g(s,v_1,v_2)\| = \max\{\sup_{s \in S}|g_1(s,v_1,v_2)|, \sup_{s \in S}|g_2(s,v_1,v_2)|\}$.

Lemma 6. The function $g(s,v_1,v_2)$ is Lipschitzian with respect to (v_1,v_2), that is for some positive constant L not depending on on s, v_1, or v_2,

$$\|g(s,v_1,v_2) - g(s,\bar{v}_1,\bar{v}_2)\| \leq L\|(v_1,v_2) - (\bar{v}_1,\bar{v}_2)\|$$

for every $s \in S$ and every pair (v_1,v_2), $(\bar{v}_1,\bar{v}_2)$.

<u>Proof.</u> Let s, (v_1,v_2), and $(\bar{v}_1,\bar{v}_2)$ be arbitrary and without loss of generality assume $g_2(s,\bar{v}_1,\bar{v}_2) \leq g_2(s,v_1,v_2)$. Suppose $\bar{a}_1 \in A_s^1$ is such that

$$-g_2(s,v_1,v_2) = \max_{a_2 \in A_s^2} \{\beta(s,\bar{a}_1,a_2)v_2 - \lambda\alpha(s,\bar{a}_1,a_2)v_1 + \gamma(s,\bar{a}_1,a_2)\} ,$$

and suppose $\bar{a}_2 \in A_s^2$ is such that

$$\max_{a_2 \in A_s^2} \{\beta(s,\bar{a}_1,a_2)\bar{v}_2 - \lambda\alpha(s,\bar{a}_1,a_2)\bar{v}_1 + \gamma(s,\bar{a}_1,a_2)\}$$

$$= \beta(s,\bar{a}_1,\bar{a}_2)\bar{v}_2 - \lambda\alpha(s,\bar{a}_1,\bar{a}_2)\bar{v}_1 + \gamma(s,\bar{a}_1,\bar{a}_2) .$$

Then $$-g_2(s,v_1,v_2)$$

$$= \min_{a_1 \in A_s^1} \max_{a_2 \in A_s^2} \{\beta(s,a_1,a_2)\bar{v}_2 - \lambda\alpha(s,a_1,a_2)\bar{v}_1 + \gamma(s,a_1,a_2)\}$$

$$\leq \max_{a_2 \in A_s^2} \{\beta(s,\bar{a}_1,a_2)\bar{v}_2 - \lambda\alpha(s,\bar{a}_1,a_2)\bar{v}_1 + \gamma(s,\bar{a}_1,a_2)\}$$

so
$$g_2(s,v_1,v_2) - g_2(s,\bar{v}_1,\bar{v}_2)$$

$$\leq \max_{a_2 \in A_s^2} \{\beta(s,\bar{a}_1,a_2)\bar{v}_2 - \lambda\alpha(s,\bar{a}_1,a_2)\bar{v}_1 + \gamma(s,\bar{a}_1,a_2)\}$$

$$- \max_{a_2 \in A_s^2} \{\beta(s,\bar{a}_1,a_2)v_2 - \lambda\alpha(s,\bar{a}_1,a_2)v_1 + \gamma(s,\bar{a}_1,a_2)\}$$

$$\leq \beta(s,\bar{a}_1,\bar{a}_2)\bar{v}_2 - \lambda\alpha(s,\bar{a}_1,\bar{a}_2)\bar{v}_1 + \gamma(s,\bar{a}_1,\bar{a}_2)$$

$$-\{\beta(s,\bar{a}_1,\bar{a}_2)v_2 - \lambda\alpha(s,\bar{a}_1,\bar{a}_2)v_1 + \gamma(s,\bar{a}_1,\bar{a}_2)\}$$

$$\leq C_1|\bar{v}_2 - v_2| + \lambda C_2|\bar{v}_1 - v_1| ,$$

where $C_1 = \max\limits_{\substack{s \in S \\ a_1 \in K_1 \\ a_2 \in K_2}} |\beta(s,a_1,a_2)|$ and $C_2 = \max\limits_{\substack{s \in S \\ a_1 \in K_1 \\ a_2 \in K_2}} |\alpha(s,a_1,a_2)|$.

Thus the desired result follows with $L = \max\{1, C_1 + \lambda C_2\}$.

In subsequent lemmas we use $v(s,u_1,u_2)$ to denote the solution of (5) on S satisfying $v(r_0,u_1,u_2) = u_1$ and $v'(r_0,u_1,u_2) = u_2$ where, of course, $v'(s,u_1,u_2) = \frac{\partial}{\partial s}v(s,u_1,u_2)$. This is not to be confused with the notation at the beginning of this section. It should be clear from the context whether the second and third arguments of $v(s,u_1,u_2)$ are boundary conditions or admissible controls.

Lemma 7. For $u_1,u_2 \in (-\infty,\infty)$ equation (5) has a unique solution

$v(s,u_1,u_2)$ on S satisfying $v(r_0,u_1,u_2) = u_1$ and $v'(r_0,u_1,u_2) = u_2$.

Proof. It suffices to show the equation

$$\frac{d}{ds}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = g(s,v_1,v_2)$$

has a unique solution on S satisfying $v_1(r_0) = u_1$ and $v_2(r_0) = u_2$ because then $v(s,u_1,u_2) = v_1(s)$. By differential equation theory and the piecewise continuity of A_g, Lemmas 5, 6 and a result in Edwards [6, pp. 153-155] imply the result.

Lemma 8. The functions $v(s,u_1,u_2)$ and $v'(s,u_1,u_2)$ are continuous, strictly increasing functions of u_2 with limits $\pm\infty$ as $u_2 \to \pm\infty$ for each fixed $s \in (r_0,r_1]$ and each fixed $u_1 \in (-\infty,\infty)$.

Proof. We first show the function $v'(s,u_1,\cdot)$ is strictly increasing. Suppose not, so that for some $u_1 \in (-\infty,\infty)$, $s_0 \in (r_0,r_1]$, and pair $u_2 < \bar{u}_2$, say, we have $v'(s_0,u_1,u_2) = v'(s_0,u_1,\bar{u}_2)$ and $v'(s,u_1,u_2) < v'(s,u_1,\bar{u}_2)$ for all $s \in [r_0,s_0)$. It follows that $v''(s,u_1,u_2) > v''(s,u_1,\bar{u}_2)$ for some $s < s_0$ in every neighborhood of s_0 and $v(s_0,u_1,u_2) < v(s_0,u_1,\bar{u}_2)$. But since $\alpha(s,a_1,a_2) > 0$ and by continuity we have $v''(s,u_1,u_2) \leq v''(s,u_1,\bar{u}_2)$ for all $s < s_0$ in some neighborhood of s_0, a contradiction. Thus $u_2 < \bar{u}_2$ must imply $v'(s,u_1,u_2) < v'(s,u_1,\bar{u}_2)$, in which case $v(s,u_1,u_2) < v(s,u_1,\bar{u}_2)$, for each $s \in (r_0,r_1]$. The continuity of $v(s,u_1,u_2)$ and $v'(s,u_1,u_2)$ with respect to u_2 follows by standard differential equation theory.

To show the limiting behavior of $v(s,u_1,u_2)$ and its derivative, it suffices to consider $u_2 \to \infty$ and $u_1 < 0$; the situation with $u_1 \geq 0$ reduces to this case and the proof with $u_2 \to -\infty$ is similar. For arbitrary $u_1 < 0$, $s_0 \in (r_0,r_1]$, and $L \in (0,\infty)$, it suffices to show $v'(s_0,u_1,u_2) \geq L$ for some $u_2 \in (-\infty,\infty)$. To do this, we consider the differential equation

$$(7) \qquad y''(s) = -C_\beta y'(s) + \lambda C_2 y(s) - C_\gamma ;$$

where $C_\beta = \max\limits_{\substack{s\in S\\ a_1\in K_1\\ a_2\in K_2}} |\beta(s,a_1,a_2)|$, $C_\gamma > \max\limits_{\substack{s\in S\\ a_1\in K_1\\ a_2\in K_2}} |\gamma(s,a_1,a_2)|$,

and $C_2 > 0$. Now if $y(s)$ is a solution of (7) with $y(r_0) = u_1 \in (-\infty,\infty)$ and $\frac{dy}{ds}(r_0) = u_2 \in (-\infty,\infty)$, then it is easy to verify that $y'(s) \to \infty$ as $u_2 \to \infty$ for each $s \in S$. In particular, with $C_2 = \underline{C} = \min\limits_{\substack{s\in S\\ a_1\in K_1\\ a_2\in K_2}} \alpha(s,a_1,a_2)$, there exists some constant $L_1 \geq L$ such that if $p \in \left[r_0, \frac{r_0+s_0}{2}\right]$, then the solution to (7) satisfying $y(p) = 0$ and $y'(p) \geq L_1$ will be such that $y'(s) \geq L$ for all $s \in [p,s_0]$. Also, with $C_2 = \overline{C} = \max\limits_{\substack{s\in S\\ a_1\in K_1\\ a_2\in K_2}} \alpha(s,a_1,a_2)$, there exists some constant u_2

such that the solution to (7) satisfying $y(r_0) = u_1$ and $y'(r_0) = u_2$ will be such that $y(\bar{s}) = 0$ for some $\bar{s} \in \left[r_0, \frac{r_0 + s_0}{2}\right]$ and $y'(s) \geq L_1$ for all $s \in [r_0, \bar{s}]$; let $\bar{y}(s)$ denote this solution on $[r_0, \bar{s}]$.

We now claim that $v'(s_0, u_1, u_2) \geq L$ because $v(s, u_1, u_2)$ is bounded below by appropriate solutions of (7). For example, suppose $\tilde{s} \in [r_0, \bar{s}]$ is such that $0 \geq v(\tilde{s}, u_1, u_2) \geq \bar{y}(\tilde{s})$ and $v'(\tilde{s}, u_1, u_2) = \bar{y}'(\tilde{s})$. Then for some $(a_1, a_2) \in A_{\tilde{s}}$

$$v''(\tilde{s}, u_1, u_2) - \bar{y}''(\tilde{s}) = [C_\beta - \beta(s, a_1, a_2)]\bar{y}'(\tilde{s})$$

$$+ \lambda[\alpha(s, a_1, a_2) - \bar{C}]v(\tilde{s}, u_1, u_2)$$

$$+ \lambda\bar{C}[v(\tilde{s}, u_1, u_2) - \bar{y}(\tilde{s})] + [C_\gamma - \gamma(\tilde{s}, a_1, a_2)] .$$

Note the last term on the right hand side is positive and the others are non-negative so $v''(\tilde{s}, u_1, u_2) > \bar{y}''(\tilde{s})$. By continuity, $v''(s, u_1, u_2) > \bar{y}''(s)$ for all s in some neighborhood of $\tilde{s}$. In particular, if we let $\tilde{s} = r_0$ then it becomes apparent that $v'(s, u_1, u_2) = \bar{y}'(s)$ is impossible with $v(s, u_1, u_2) \leq 0$ for $s \in (r_0, \bar{s}]$. Thus $v'(s, u_1, u_2) > \bar{y}'(s)$ for each such s and there exists some $p \in (r_0, \bar{s}]$ such that $v(p, u_1, u_2) = 0$ and $v'(p, u_1, u_2) \geq L_1$.

Now let $\underline{y}(s)$ be the solution to (7) with $\underline{y}(p) = 0$, $\underline{y}'(p) = v'(p, u_1, u_2)$, and $C_2 = \underline{C}$ and note that $\underline{y}'(s) \geq L$ for all $s \in [p, s_0]$. By comparing $v(s, u_1, u_2)$ with $\underline{y}(s)$ as we did with $\bar{y}(s)$, we conclude the desired result.

Lemma 9. For fixed u_1 the function

$$v(r_1,u_1,u_2) - \int_S v(s,u_1,u_2)\,d\mu_1(s)$$

is continuous and strictly increasing in u_2 and diverges to $\pm\infty$ as $u_2 \to \pm\infty$.

Proof. Let $u_2 < u_2'$ and $\bar{s} \in [r_0,r_1]$ be arbitrary. By Lemma 8, $v'(s,u_1,u_2)$ is increasing in u_2, so $v(r_1,u_1,u_2') - v(\bar{s},u_1,u_2') > v(r_1,u_1,u_2) - v(\bar{s},u_1,u_2)$. The function in this lemma is just a convex combination of the right hand side of this inequality, so by this inequality this function is increasing in u_2.

Since $v'(s,u_1,u_2) \to \pm\infty$ as $u_2 \to \pm\infty$, we have for any $s \in (r_0,r_1)$ that $v(r_1,u_1,u_2) - v(\bar{s},u_1,u_2) \to \pm\infty$ as $u_2 \to \pm\infty$. Thus, by the convex combination argument the function in this lemma has the same limits. Continuity follows from the continuity of $v(s,u_1,u_2)$.

Lemma 10. Let $\beta(s)$, $\alpha(s)$ and $\gamma(s)$ be measurable real-valued functions on S with $|\beta(s)| \le C_\beta < \infty$, $\alpha(s) \ge C_2 > 0$, and $\gamma(s) \ge 0$, and suppose for some $\bar{s} \in [r_0,r_1)$, $u_1 > 0$, and $u_2 \ge 0$, the function $v(s)$ is a solution to

$$v''(s) = \beta(s)v\ (s) + \alpha(s)v(s) + \gamma(s)$$

satisfying $v(\bar{s}) = u_1$ and $v'(\bar{s}) = u_2$.

Then for all $s \in (\bar{s}, r]$ we have $v(s) > 0$ and $v'(s) > 0$.

<u>Proof</u>. Suppose there is a smallest $s_0 \geq \bar{s}$ such that $v'(s_0) = 0$. Then $v(s_0) > 0$, and by continuity, $v''(s) > 0$ in some neighborhood of s_0. Hence $v'(s) < 0$ for all large enough $s < s_0$, contradicting $u_2 \geq 0$ and the definition of s_0.

<u>Proof of Theorem 3</u>. Denote

$$N_j = \theta_j \int_S v_j(s) d\mu_j(s) + \sigma_j \gamma_j + \kappa_j \lambda_j \; ; \qquad j = 0,1 .$$

By Lemma 7 it suffices to show that $v(s, u_1, u_2)$, the unique solution of (5) with $v(r_0, u_1, u_2) = u_1$ and $v'(r_0, u_1, u_2) = u_2$, satisfies

$$(8) \qquad (\lambda\sigma_j + \theta_j + \kappa_j) v(r_j, u_1, u_2) - \theta_j \int_S v(s, u_1, u_2) d\mu_j(s)$$

$$-(-1)^j \pi_j v'(r_j, u_1, u_2) = N_j \qquad j = 0,1,$$

for unique values of u_1 and u_2. There are two cases.

Case 1: $\theta_0 = \pi_0 = 0$

By (8), $u_1 = N_0/(\lambda\sigma_0 + \kappa_0)$. By Lemmas 8 and 9, the left hand side of (8) for $j = 1$ increases continuously and strictly from $-\infty$ to ∞ as u_2 increases from $-\infty$ to ∞. Hence (8) for $j = 1$ is satisfied by a unique value of u_2.

Case 2: $\theta_0 + \pi_0 > 0$

For $P \varepsilon (-\infty, \infty)$ denote

$$f_P(s,v_1,v_2) = -\min_{a_1 \varepsilon A_s^1} \max_{a_2 \varepsilon A_s^2} \{\beta(s,a_1,a_2)v_2 - \lambda\alpha(s,a_1,a_2)v_1 + P\gamma(s,a_1,a_2)\}$$

and let $v_P(s,u_1,u_2)$ denote the unique solution to

$$\text{(9)} \qquad v_P''(s,u_1,u_2) = f_P(s,v_P(s,u_1,u_2),\ v_P'(s,u_1,u_2))$$

and

$$\text{(10)} \qquad v_P(r_0,u_1,u_2) = u_1 \ ; \quad v_P'(r_0,u_1,u_2) = u_2 \ .$$

Then by differential equation theory $v_P(s,u_1,u_2)$ and $v_P'(s,u_1,u_2)$ are continuous in (P,s,u_1,u_2). We seek to show that $v_1(s,u_1,u_2)$ satisfies boundary condition (8) for a unique choice of the pair u_1,u_2.

We can rewrite (8) for $j = 0$ and general P as

$$\text{(11)} \qquad \theta_0 \int_S v_P(s,u_1,u_2)\,d\mu_0(s) + \pi_0 u_2 = \phi(u_1)$$

where

$$\text{(12)} \qquad \phi(u_1) = (\lambda\sigma_0 + \theta_0 + \pi_0)u_1 - N_0 \ .$$

We first show that for each $u_1 \varepsilon (-\infty,\infty)$ and $P \varepsilon (-\infty,\infty)$ there exists a unique $u_2 = u_2(P,u_1)$ satisfying (11). But this follows from Lemma 8, because then the left hand side of (11) is continuous and strictly increasing in u_2 with limits $\pm\infty$ as $u_2 \to \pm\infty$. Note that, since both sides of (11) are continuous in (P,u_1,u_2), the function $u_2(P,u_1)$ is continuous in (P,u_1).

It remains to show that $v(s,u_1) \equiv v_1(s,u_1,u_2(1,u_1))$ satisfies (8) with $j = 1$ for a unique value of u_1, that is, there is a unique u_1 for which

$$(13)\quad (\lambda\sigma_1 + \theta_1 + \kappa_1)v(r_1,u_1) - \theta_1 \int_S v(s,u_1)d\mu_1(s) + \pi_1 v'(r_1,u_1) = N .$$

We first show that some $u_1 \in (-\infty,\infty)$ satisfies (13). Since the left hand side of (13) is continuous in u_1, it suffices to show that the left hand side of (13) diverges to $\pm\infty$ as $u_1 \to \pm\infty$. We discuss only the case $u_1 \to +\infty$ since the other is similar.

We show this result by considering the limit of $Pv(s,P^{-1})$ as $P \downarrow 0$. To this end, for $P > 0$ denote $u(P) = Pu_2(1,P^{-1})$ and $\psi(P) = P\phi(P^{-1})$. Now $Pv_1(s,P^{-1},u_2) = v_P(s,1,Pu_2)$, so $Pv(s,P^{-1}) = v_P(s,1,u(P))$. In view of this and (11), $u_2 = u(P)$ is the unique number satisfying

$$(14)\quad \theta_0 \int_S v_P(s,1,u_2)d\mu_0(s) + \pi_0 u_2 = \psi(P) .$$

Since $\psi(P)$ has a limit as $P \downarrow 0$, which we denote by $\psi(0)$, equation (14) has a unique solution $\bar{u}_2$ for $P = 0$. Since $v_P(s,1,u_2)$ and $\psi(P)$ are continuous in (P,s,u_2), it follows that $u(P)$ is continuous in P and has the limit $\bar{u}_2$ as $P \downarrow 0$; we denote $u(0) = \bar{u}_2$. In summary, $Pv(s,P^{-1}) \to v_0(s,1,u(0))$ as $P \downarrow 0$.

We are now in a position to show that the left hand side of (13) diverges to $\pm\infty$ as $u_1 \to \pm\infty$. If $\frac{dv_0}{ds}(r_0,1,u(0)) \geq 0$, then Lemma 10 applies and $v'(s,1,u(0)) > 0$ for each $s \in (r_0,r_1]$. On the other hand, if $v_0'(r_0,1,u(0)) < 0$, then by (14)

$$\theta_0 \int_S v_0(s,1,u(0))d\mu_0(s) = \lambda\sigma_0 + \kappa_0 + \theta_0 - \pi_0 v_0'(r_0,1,u(0)) \geq \theta_0 .$$

Now $\theta_0 > 0$; for if not then $\pi_0 > 0$ and by (14) $v_0'(r_0,1,u(0)) = (\lambda\sigma_0 + \kappa_0)/\pi_0 > 0$, a contradiction. Thus for at least one $s_1 \in (r_0,r_1)$ where $d\mu_0(s_1) > 0$ we must have $v_0(s_1,1,u(0)) \geq 1 = v_0(r_0,1,u(0))$. It follows for some $s_0 \in [r_0,s_1]$ that $v_0(s_0,1,u(0)) > 0$ and $v_0'(s_0,1,u(0)) \geq 0$. Applying Lemma 10, we conclude for all $s \in (s_0,r_1]$ that $v_0(s,1,u(0)) > 0$ and $v_0'(s,1,u(0)) > 0$.

Let Y denote the left hand side of (13) with $v_0(s,1,u(0))$ substituted for $v(s,u_1)$. We have by the preceding arguments that $v_0(r_1,1,u(0)) > 0$ and $v'(r_1,1,u(0)) > 0$. Moreover, for any $s \in [r_0,r_1)$ we have $v_0(r_1,1,u(0)) > v_0(s,1,u(0))$, so if $\theta_1 > 0$ then

$$\theta_1\left[v_0(r_1,1,u(0)) - \int_S v_0(s,1,u(0))d\mu_1(s)\right] > 0 ,$$

in which case Y 0. Letting $u_1 \to \infty$ in the left hand side of (13), we have

$$\lim_{u_1\to\infty} \{(\lambda\sigma_1 + \theta_1 + \kappa_1)v(r_1,u_1) - \theta_1 \int_S v(s,u_1)d\mu_1(s) + \pi_1 v'(r_1,u_1)\}$$

$$= \lim_{P\downarrow 0} \frac{1}{P}\{(\lambda\sigma_1 + \theta_1 + \kappa_1)v_p(s,1,u(P)) - \theta_1 \int_S v_p(s,1,u(P))d\mu_1(s)$$

$$+ \pi_1 v_p'(r_1,1,u(P))\}$$

$$= \lim_{P\downarrow 0} \frac{Y}{P} = +\infty \ .$$

Thus, by the remarks following equation (13) there exists some $u_1 \varepsilon (-\infty,\infty)$ which satisfies (13); it remains to show this u_1 is unique.

Suppose there exist two numbers $C_0 < C_1$ and corresponding solutions $v_0(s)$ and $v_1(s)$ of (5), (6) such that $v_0(r_0) = C_0$ and $v_1(v_0) = C_1$. Let the Borel measurable function $a_1(s)$ from S into K_1 be such that $a_1(s) \varepsilon A_s^1$ and

$$\min_{a_1 \varepsilon A_s^1} \ \max_{a_2 \varepsilon A_s^2} \{\beta(s,a_1, a_2)v_0'(s) - \lambda\alpha(s,a_1,a_2)v_0(s) + \gamma(s,a_1,a_2)\}$$

$$= \max_{a_2 \ A_s^2} \{\beta(s,a_1(s),a_2)v_0'(s) - \lambda\alpha(s,a_1(s),a_2)v_0(s) + \gamma(s,a_1(s),a_2)\}$$

for each $s \varepsilon S$. Let the Borel measurable function $a_2(s)$ from S into K_2 be such that $a_2(s) \varepsilon A_s^2$ and

$$\max_{a_2 \varepsilon A_s^2} \{\beta(s,a_1(s),a_2)v_1'(s) - \lambda\alpha(s,a_1(s),a_2)v_1(s) + \gamma(s,a_1(s),a_2)\}$$

$$= \beta(s,a_1(s),a_2(s))v_1'(s) - \lambda\alpha(s,a_1(s),a_2(s))v_1(s) + \gamma(s,a_1(s),a_2(s)$$

for each $s \varepsilon S$. Then

$$0 = v_1''(s) + \min_{a_1 \varepsilon A_s^1} \max_{a_2 \varepsilon A_s^2} \{\beta(s,a_1,a_2)v_1'(s) - \lambda\alpha(s,a_1,a_2)v_1(s) + \gamma(s,a_1,a_2)\}$$

$$\leq v_1''(s) + \max_{a_2 \varepsilon A_s^2} \{\beta(s,a_1(s),a_2)v_1'(s) - \lambda\alpha(s,a_1(s),a_2)v_1(s) + \gamma(s,a_1(s),a_2)\}$$

$$= v_1''(s) + \beta(s,a_1(s),a_2(s))v_1'(s) - \lambda\alpha(s,a_1(s),a_2(s)v_1(s) + \gamma(s,a_1(s),a_2(s))$$

and similarly,

$$v_0''(s) + \beta(s,a_1(s),a_2(s))v_0'(s) - \lambda\alpha(s,a_1(s),a_2(s))v_0(s) + \gamma(s,a_1(s),a_2(s)) \leq 0 .$$

Defining the Borel measurable function

$$\psi(s) = \frac{d^2}{ds^2}(v_1 - v_0)(s) + \beta(s,a_1(s),a_2(s))\frac{d}{ds}(v_1 - v_0)(s) - \lambda\alpha(s,a_1(s),a_2(s))(v_1 - v_0)(s) ;$$

we see that $\psi(s) \geq 0$. Letting $v(s) = v_1(s) - v_0(s)$, we see that $v(s)$ is a solution to

$$v''(s) = -\beta(s,a_1(s),a_2(s))v'(s) + \lambda\alpha(s,a_1(s),a_2(s))v(s) + \psi(s)$$

satisfying (by subtracting boundary conditions (6))

$$(15)\qquad (\lambda\sigma_j + \theta_j + \kappa_j)v(r_j) - \theta_j \int_S v(s)d\mu_j(s) - (-1)^j \pi_j v'(r_j) = 0 .$$

$$j = 0,1,$$

as well as $v(r_0) = v(r_0) = C_1 - C_0$.

It remains to show that $v(s)$ cannot simultaneously satisfy all three of these boundary conditions. Assume $v(r_0) = C_1 - C_0$ and (15) holds for $j = 0$. Let $s_0 = \min\{s \in S | v(s) = C_1 - C_0$ and $v'(s) \geq 0\}$. Note that s_0 exists because if $v'(r_0) \geq 0$ then $s_0 = r_0$, whereas if $v'(r_0) < 0$, then by (15)

$$\theta_0 \int_S v(s)\, d\mu_0(s) = (\lambda\sigma_0 + \theta_0 + \kappa_0)v(r_0) - \pi_0 v'(r_0) \geq \theta_0 v(r_0)$$

so $\theta_0 > 0$ and for some $s_1 \in (r_0, r_1]$ with $d\mu_0(s_1) > 0$ we have $v(s_1) \geq v(r_0)$ in which case $s_0 \in (r_0, s_1]$. By Lemma 10 we have $v(s) > 0$ and $v'(s) > 0$ for all $s \in [s_0, r_1]$. In particular, the left hand side of (15) is positive for $j = 1$ and Theorem 3 is proved.

The following theorem provides a necessary and sufficient (saddlepoint) condition for an admissible control to be a solution to the zero sum, two-person game. We now revert to the original notation, where $v(s, a_1, a_2)$ denotes the expected discounted cost of a process corresponding to the control $a = (a_1, a_2) \in M$.

<u>Theorem 11</u>. Let $v(s)$ be the unique solution of (5), (6). A control $\hat{a} = (\hat{a}_1, \hat{a}_2) \in M$ is optimal if and only if

$$(16)\qquad \max_{a_2 \varepsilon A_s^2}\{d(s,\hat{a}_1(s),a_2)^{-1}[b(s,\hat{a}_1(s),a_2)\hat{v}'(s) - \lambda\hat{v}(s) + c(s,\hat{a}_1(s),a_2)]\}$$

$$= d(s,\hat{a}_1(s),\hat{a}_2(s))^{-1}[b(s,\hat{a}_1(s),\hat{a}_2(s))\hat{v}'(s) - \lambda\hat{v}(s) + c(s,\hat{a}_1(s),\hat{a}_2(s))]$$

$$= \min_{a_1 \varepsilon A_s^1}\{d(s,a_1,\hat{a}_2(s))^{-1}[b(s,a_1,\hat{a}_2(s))\hat{v}'(s) - \lambda\hat{v}(s) + c(s,a_1,\hat{a}_2(s))]\}$$

for every $s \varepsilon S$ which is a continuity point of $\hat{a}$ and, for $j = 0$ and $j = 1$, $\sigma_j > 0$ implies

$$(17)\qquad \max_{a_2 \varepsilon A_{r_j}^2} c(r_j,\hat{a}_1(r_j),a_2) = c(r_j,\hat{a}_1(r_j),\hat{a}_2(r_j))$$

$$= \min_{a_1 \varepsilon A_{r_j}^1} c(r_j,a_1,\hat{a}_2(r_j))$$

Moreover, if $\hat{a}$ is optimal then $\hat{v}(s) = v(s,\hat{a}_1,\hat{a}_2)$ is the value of the game.

Proof. Suppose (16) and (17) are true. By the theory of saddlepoints we have

$$\min_{a_1 \varepsilon A_s^1} \max_{a_2 \varepsilon A_s^2}\{d(s,a_1,a_2)^{-1}[b(s,a_1,a_2)\hat{v}'(s) - \lambda\hat{v}(s) + c(s,a_1,a_2)]\}$$

$$= d(s,\hat{a}_1(s),\hat{a}_2(s))^{-1}[b(s,\hat{a}_1(s),\hat{a}_2(s))\hat{v}'(s) - \lambda\hat{v}(s) + c(s,\hat{a}_1(s),\hat{a}_2(s))]$$

and, if $\sigma_j > 0$, $\min_{a_1 \varepsilon A^1_{r_j}} \max_{a_2 \varepsilon A^2_{r_j}} c(r_j, a_1, a_2) = c(r_j, \hat{a}_1(r_j), \hat{a}_2(r_j))$.

Substituting these in (5) and (6), we see that $\hat{v}(s)$ is also the unique solution of (1) and (2), that is, $\hat{v}(s) = v(s, \hat{a}_1, \hat{a}_2)$.

With $\hat{a}_2(s)$ fixed, in a similar manner we see that $\hat{v}(s)$ is the unique solution of (5), (6) of Chapter II, that is, $\hat{v}(s)$ is the minimal expected discounted cost for an ordinary optimal control problem involving the control $a_1(s)$. In view of (16) and (17), we have by Theorem 6 of Chapter II that $v(s, \hat{a}_1, \hat{a}_2) \leq v(s, a_1, \hat{a}_2)$ for each $a_1 \varepsilon M_1$ and each $s \varepsilon S$. Similarly, $v(s, \hat{a}_1, a_2) \leq v(s, \hat{a}_1, \hat{a}_2)$ for all $a_2 \varepsilon M_2$. Hence $\hat{a}$ is optimal and $\hat{v}(s)$ is the value of the game.

Conversely, suppose $\hat{a}$ is an optimal control. First we'll show that

$$
\begin{aligned}
& d(s, a_1(s), \hat{a}_2(s))^{-1}[b(s, \hat{a}_1(s), \hat{a}_2(s))v'(s, \hat{a}_1, \hat{a}_2) - \lambda v(s, \hat{a}_1, \hat{a}_2) \\
& \qquad\qquad + c(s, \hat{a}_1(s), \hat{a}_2(s))] \\
& = \max_{a_2 \varepsilon A^2_s} \{d(s, a_1(s), a_2)^{-1}[b(s, a_1(s), a_2)v'(s, \hat{a}_1, \hat{a}_2) - \lambda v(s, \hat{a}_1, \hat{a}_2) \\
& \qquad\qquad + c(s, a_1(s), a_2)]\} \\
& \geq \min_{a_1 \varepsilon A^1_s} \max_{a_2 \varepsilon A^2_s} \{d(s, a_1, a_2)^{-1}[b(s, a_1, a_2)v'(s, a_1, \hat{a}_2) - \lambda v(s, \hat{a}_1, \hat{a}_2) \\
& \qquad\qquad + c(s, a_1, a_2)]\} \\
& \geq \max_{a_2 \varepsilon A^2_s} \min_{a_1 \varepsilon A^1_s} \{d(s, a_1, a_2)^{-1}[b(s, a_1, a_2)v'(s, \hat{a}_1, \hat{a}_2) - \lambda v(s, a_1, \hat{a}_2) \\
& \qquad\qquad + c(s, a_1, a_2)]\}
\end{aligned}
$$

$$\geq \min_{a_1 \varepsilon A_s^1} \{d(s,a_1,\hat{a}_2(s))^{-1}[b(s,a_1,\hat{a}_2(s))v'(s,\hat{a}_1,\hat{a}_2) - \lambda v(s,\hat{a}_1,\hat{a}_2) + c(s,a_1,\hat{a}_2(s))]\}$$

$$= d(s,\hat{a}_1(s),\hat{a}_2(s))^{-1}[b(s,\hat{a}_1(s),\hat{a}_2(s))v'(s,\hat{a}_1,\hat{a}_2) - \lambda v(s,\hat{a}_1,\hat{a}_2) + c(s,\hat{a}_1(s),\hat{a}_2(s))] .$$

Now $v(s,\hat{a}_1,\hat{a}_2) = \inf_{a_1 \varepsilon M_1} v(s,a_1,\hat{a}_2)$, so by Theorem 6 of Chapter II, the last inequality is true; similarly, the first one is true. The inequalities are true by saddlepoint theory, so all are equalities. Similarly, if $\sigma_j > 0$, then

$$c(r_j,\hat{a}_1(r_j),\hat{a}_2(r_j)) = \max_{a_2 \varepsilon A_{r_j}^2} c(r_j,\hat{a}_1(r_j),a_2)$$

$$= \min_{a_1 \varepsilon A_{r_j}^1} \max_{a_2 \varepsilon A_{r_j}^2} c(r_j,a_1,a_2) = \min_{a_1 \varepsilon A_{r_j}^1} c(r_j,a_1,\hat{a}_2(r_j)) .$$

Substituting these equalities into (1) and (2), we see that $v(s,\hat{a}_1,\hat{a}_2)$ is the unique solution of (5) and (6), that is, $v(s,\hat{a}_1,\hat{a}_2) = v(s)$. Substituting $\hat{v}(s)$ for $v(s,\hat{a}_1,\hat{a}_2)$ in the above equalities yields (16), and Theorem 11 is proved.

A diffusion process two-person, zero sum game problem can be solved in principle as follows. First, obtain the solution $v(s)$ to (5) and (6). Second, consider the map Γ from S into $K_1 \times K_2$ such that $(a_1,a_2) \varepsilon \Gamma(s)$ if and only if $a_1 \varepsilon A_s^1$, $a_2 \varepsilon A_s^2$, and (a_1,a_2)

is a saddlepoint of $d(s,a_1,a_2)^{-1}[b(s,a_1,a_2)v'(s) - \lambda v(s) + c(s,a_1,a_2)]$. If $\sigma_j > 0$, $j = 0,1$, then redefine $\Gamma(r_j)$ so that $(a_1,a_2) \in \Gamma(r_j)$ if and only if $a_1 \in A^1_{r_j}$, $a_2 \in A^2_{r_j}$, and (a_1,a_2) is a saddlepoint of $c(r_j,a_1,a_2)$. Note that $\Gamma(s) = \phi$ is possible for some $s \in S$, in which case the game is without solution. On the other hand, if $\Gamma(s) \neq \phi$ for each $s \in S$, then $v(s)$ is the value of the game, even if it cannot be attained. Finally, endeavor to choose a piecewise continuous function $a(s)$ such that $a(s) \in \Gamma(s)$ for each $s \in S$.

The following result is a sufficient condition for (16) and (17) to be satisfied by $v(s)$ and some Borel measurable control $a(s)$, that is, for the map mentioned above $\Gamma(s) \neq \phi$ for each $s \in S$. The real-valued function $h(z)$ on the compact, convex set $C \subset E^n$ is said to be quasiconvex if $\{z \in C | h(z) \leq \alpha\}$ is convex for each $\alpha \in E$. This function is quasiconcave if $-h(z)$ is quasiconvex. Corollary 13 is an immediate consequence of Theorem 12 which, in turn, follows easily from a minimax theorem by Sion [20].

Theorem 12. Let $v(s)$ be the unique solution of (5), (6) and suppose A^i_s is convex for each $s \in S$, $i = 1,2$. Then there exists some Borel measurable control $a(s) = (a_1(s),a_2(s))$, with $a_1(s) \in A^1_s$ and $a_2(s) \in A^2_s$ for each $s \in S$, which satisfies (16) and (17) provided

$$d(s,a_1,a_2)^{-1}[b(s,a_1,a_2)v'(s) - \lambda v(s) + c(s,a_1,a_2)]$$

and $\sigma_j c(r_j,a_1,a_2)$, $j = 0,1$, are quasiconvex in $a_1 \in A^1_s$ for each $a_2 \in A^2_s$ and $s \in S$ and are quasiconcave in $a_2 \in A^2_s$ for each

$a_1 \in A_s^1$ and $s \in S$.

Corollary 13. Let $v(s)$ be the unique solution of (5), (6) and suppose A_s^i is convex for each $s \in S$, $i = 1,2$. Suppose $d(s,a_1,a_2)$ is constant with respect to (a_1,a_2), $b(s,a_1,a_2)$ is affine with respect to (a_1,a_2), $c(s,a_1,a_2)$ is convex in a_1 for each $a_2 \in A_s^2$, and $c(s,a_1,a_2)$ is concave in a_2 for each $a_1 \in A_s^1$, all for each $s \in S$. Then (16) and (17) are satisfied by some Borel measurable control $(a_1(s),a_2(s))$ with $a_1(s) \in A_s^1$ and $a_2(s) \in A_s^2$.

Example.

$0 < r_0 < r_1$	$d(s,a_1,a_2) = A > 0$
$A_s^1 = \{a_1 \in E \mid \lvert a_1\rvert \le z_1 s\},\ z_1 > 0$	$b(s,a_1,a_2) = a_1a_2/s$
$A_s^2 = \{a_2 \in E \mid \lvert a_2\rvert \le z_2 s\},\ z_2 > 0$	$c(s,a_1,a_2) = C$

r_0 boundary condition. $v'(r_0) = 0$ (reflection)

r_1 boundary condition; $v(r_1) = \lambda_1$ (absorption with cost λ_1)

For any value of s, $v(s)$, or $v'(s)$ we have

$$\min_{a_1 \in A_s^1} \max_{a_2 \in A_s^2} A^{-1}\left[\frac{a_1a_2}{s}v'(s) - \lambda v(s) + C\right]$$

$$= \max_{a_2 \in A_s^2} \min_{a_1 \in A_s^1} A^{-1}\left[\frac{a_1a_2}{s}v'(s) - \lambda v(s) + C\right]$$

$$= A^{-1}[C - \lambda v(s)] ,$$

so $a(s) = (0,0)$ is the optimal control for all $s \in S$. Therefore, the value of the game is given by (1), (2) to be $v(s) = C_1 e^{tx} + C_2 e^{-tx} + C/\lambda$, where $t = \sqrt{\lambda/A}$, $C_1 = e^{tr_1}(\lambda_1 - C/\lambda)/(e^{2tr_1} + e^{2tr_0})$, and $C_2 = e^{tr_1}(\lambda_1 - C/\lambda)/(e^{2t(r_1 - r_0)} + 1)$.

4. The Zero Sum Problem with Undiscounted Costs.

The zero sum, two-person diffusion process game problem with undiscounted costs will be one of two types, depending on whether the boundary conditions are conservative or non-conservative. The results in this section parallel those of Section 3, and, consequently, they will brief. The conservative case will be treated in the second half of this section. For the purposes of this section, the boundary conditions are said to be non-conservative if at least one boundary is absorbing and neither boundary is purely adhesive, that is,

$$\kappa_0 + \kappa_1 > 0, \quad \kappa_j + \pi_j + \theta_j > 0 \qquad j = 0,1 .$$

Let $v(s,a_1,a_2) = v(s)$ denote the expected undiscounted cost of a non-conservative process corresponding to the admissible control $a = (a_1,a_2) \in M$. Then $v(s,a_1,a_2)$ will be the unique soltuion of (1), (2) with $\lambda = 0$. The control $a \in M$ is said to be optimal if for all $a_1 \in M_1$, all $a_2 \in M_2$, and all $s \in$ ^ we have

$$v(s,\hat{a}_1,a_2) \le v(s,\hat{a}_1,\hat{a}_2) \le v(s,a_1,\hat{a}_2) ,$$

in which case $v(s,\hat{a}_1,\hat{a}_2)$ is said to be the value of the game. It will subsequently be proved that the value of a game, if it exists, is provided by the following result whose proof is a generalization of one by Mandl [16, pp. 158-167].

<u>Theorem 14</u>. With non-conservative boundary conditions, there exists a unique solution $v(s)$ to

$$(18) \quad v''(s) + \min_{a_1 \varepsilon A_s^1} \max_{a_2 \varepsilon A_s^2} \{d(s,a_1,a_2)^{-1}[b(s,a_1,a_2)v'(s) + c(s,a_1,a_2)]\} = 0$$

satisfying

$$(19) \qquad (\theta_j + \kappa_j)v(r_j) - \theta_j \int_S (v(s) + \nu_j(s))d\mu_j(s)$$

$$-(-1)^j \pi_j v'(r_j) - \sigma_j \gamma_j - \kappa_j \lambda_j = 0 \qquad j = 0,1,$$

where $\qquad \gamma_j = \min_{a_1 \varepsilon A_{r_j}^1} \max_{a_2 \varepsilon A_{r_j}^2} c(r_j,a_1,a_2) , \qquad j = 0,1 .$

<u>Proof</u>. Lemma 7 does not depend on $\lambda > 0$, so for every $u_1,u_2 \in (-\infty,\infty)$ equation (18) has a unique solution $v(s)$ satisfying $v(r_0) = u_1$ and $v'(r_0) = u_2$. For fixed u_1 and u_2 denote $w(s,u_2) = v'(s)$ and note that $w(s,u_2)$ is independent of u_1 since it is the solution of a first order differential equation under the initial condition

$w(r_0,u_2) = u_2$. Writing for $j = 0,1$

$$N_j = \theta_j \int_S \nu_j(s)d\mu_j(s) + \sigma_j\gamma_j + \kappa_j\lambda_j \, .$$

we have that (19) is equivalent to

$$\text{(20)} \qquad \kappa_0 u_1 - \theta_0 \int_S \int_{r_0}^{s} w(t,u_2)dtd\mu_0(s) - \pi_0 u_2 = N_0 \, ,$$

$$\kappa_1 u_1 + (\theta_1 + \kappa_1) \int_{r_0}^{r_1} w(t,u_2)dt - \theta_1 \int_S \int_{r_0}^{s} w(t,u_2)dtd\mu_1(s)$$

$$+ \pi_1 w(r_1,u_2) = N_1 \, .$$

Eliminating u_1 from (20), we obtain the equation for u_2 :

$$\text{(21)} \qquad \kappa_0(\theta_1 + \kappa_1) \int_{r_0}^{r_1} w(t,u_2)dt + \kappa_1\theta_0 \int_S \int_{r_0}^{s} w(t,u_2)dtd\mu_0(s)$$

$$+ \kappa_1\pi_0 u_2 + \kappa_0\pi_1 w(r_1,u_2) - \kappa_0\theta_1 \int_S \int_{r_0}^{s} w(t,u_2)dtd\mu_1(s) = \kappa_0 N_1 - \kappa_1 N_0 \, .$$

It remains to show that (21) is solved by a unique value of u_2, since then u_1 can be obtained from (20). By Lemma 8, $w(s,u_2)$ is continuous and strictly increasing in u_2 and $w(s,u_2) \to \pm\infty$ as $u_2 \to \pm\infty$, in which case the left hand side of (21) has these same properties (see Mandl [16, p. 163]). Hence (21) has a unique solution and Theorem 14 is proved.

Theorem 15. With undiscounted costs and non-conservative boundary conditions, let $v(s)$ be the unique solution of (18), (19). A control $a = (a_1, a_2) \in M$ is optimal if and only if

(22) $$\max_{a_2 \in A_s^2} \{d(s,a_1(s),a_2)^{-1}[b(s,a_1(s),a_2)v'(s) + c(s,a_1(s),a_2)]\}$$

$$= d(s,a_1(s),a_2(s))^{-1}[b(s,a_1(s),a_2(s))v'(s) + c(s,a_1(s),a_2(s))]$$

$$= \min_{a_1 \in A_s^1} \{d(s,a_1,a_2(s))^{-1}[b(s,a_1,a_2(s))v'(s) + c(s,a_1,a_2(s))]\}$$

for each $s \in S$ which is a continuity point of $a(s)$, and, for $j = 0$ and $j = 1$, $\sigma_j > 0$ implies

(23) $$\max_{a_2 \in A_{r_j}^2} c(r_j,a_1(r_j),a_2) = c(r_j,a_1(r_j),a_2(r_j))$$

$$= \min_{a_1 \in A_{r_j}^1} c(r_j,a_1,a_2(r_j)) .$$

Moreover, if $a(s)$ is optimal, then $v(s) = v(s,a_1,a_2)$ is the value of the game.

This proof is essentially identical to that for Theorem 11, so it will be omitted. A diffusion process zero sum, two-person game problem in the undiscounted cost, non-conservative process case can be solved, in principle, in the same manner as with the discounted cost case. Moreover, there exist sufficient conditions analogous to those of Theorem 12 and Corollary 13 for equations (22) and (23) to be satisfied by some Borel measurable control $a(s)$.

Example.

$S = [0,1]$ $\qquad$ $d(s,a_1,a_2) = A > 0$

$A_s^1 = [-Y,Y] \subset E, \quad Y > 0$ $\qquad$ $b(s,a_1,a_2) = a_1 a_2$

$A_s^2 = [-Z,Z] \subset E, \quad Z > 0$ $\qquad$ $c(s,a_1,a_2) = C$

r_0 boundary condition: $v(r_0) = \lambda_0$ (absorption with cost λ_0) .

r_1 boundary condition: $v'(r_1) = 0$ (reflection) .

For any value of s or $v'(s)$ we have

$$\min_{a_1 \epsilon A_s^1} \max_{a_2 \epsilon A_s^2} \{A^{-1}[a_1 a_2 v'(s) + C]\}$$

$$= \max_{a_2 \epsilon A_s^2} \min_{a_1 \epsilon A_s^1} \{A^{-1}[a_1 a_2 v'(s) + C]\} = A^{-1}C ,$$

so $a(s) = (0,0)$ is the optimal control for all $s \in S$. Therefore, the value of the game is given by (1) and (2) to be $v(s) = -\frac{1}{2}(C/A)s^2 + (C/A)s + \lambda_0$.

We now discuss the other type of undiscounted cost problem, the conservative case. For the purposes of this section, the boundary conditions are said to be conservative if neither boundary is absorbing and at least one boundary is not purely adhesive, that is,

$$\kappa_0 = \kappa_1 = 0 , \qquad \pi_0 + \theta_0 + \pi_1 + \theta_1 > 0 .$$

Let $\Theta(a_1,a_2) = \Theta$ denote the mean cost per unit time of such a process corresponding to the admissible control $a = (a_1,a_2) \in M$. Then $\Theta(a_1,a_2)$ is the unique number to which there exists a solution to (3) and (4). The control $\hat{a} \in M$ is said to be optimal if for all $a_1 \in M_1$ and all $a_2 \in M_2$ we have

$$\Theta(\hat{a}_1,a_2) \leq \Theta(\hat{a}_1,\hat{a}_2) \leq \Theta(a_1,\hat{a}_2) ,$$

in which case $\Theta(\hat{a}_1,\hat{a}_2)$ is said to be the value of the game. The following result characterizes the value of a game.

<u>Theorem 16</u>. With conservative boundary conditions there exists a unique number Θ such that the equation

$$(24) \qquad w'(s) + \min_{a_1 \in A_s^1} \max_{a_2 \in A_s^2} \{d(s,a_1,a_2)^{-1}[b(s,a_1,a_2)w(s) - \Theta + c(s,a_1,a_2)]\} = 0$$

has a solution $w(s)$ satisfying

$$(25) \qquad \varepsilon_j \left[\int_S \int_{r_j}^{s} w(y)dy + \kappa_j(s) \right] d\mu_j(s) + (-1)^j \pi_j w(r_j) + \theta_j(\gamma_j - \Theta) = 0 ,$$

$$j = 0, 1,$$

where $\gamma_j = \min_{a_1 \in A_{r_j}^1} \max_{a_2 \in A_{r_j}^2} c(r_j,a_1,a_2)$.

Proof. This proof is rather similar to that for Theorem 3, so it will only be sketched. By Lemma 7 for every u_2, $\Theta \in (-\infty,\infty)$ there exists a unique solution $w(s,u_2,\Theta)$ to (24) satisfying $w(r_0,u_2,\Theta) = u_2$. By Lemma 8, $w(s,u_2,\Theta)$ is continuous and strictly increasing in u_2 and $w(s,u_2,\Theta) \to \pm\infty$ as $u_2 \to \pm\infty$. It follows that the left hand side of (25) with $w(s,u_2,\Theta)$ substituted for $w(s)$ is continuous and strictly increasing (decreasing) in u_2 and diverges to $\pm\infty$ $(\mp\infty)$ as $u_2 \to \pm\infty$ for $j = 0$ $(j = 1)$. Thus, if boundary r_j is purely adhesive, then $\Theta = \gamma_j$ and u_2 can be determined uniquely from the other boundary condition.

On the other hand, if neither boundary is purely adhesive, then to every Θ there exists a unique number $u_2 = u_2(\Theta)$ such that $w(s,\Theta) \equiv w(s,u_2(\Theta),\Theta)$ satisfies (25) for $j = 0$. It remains to show that $w(s,\Theta)$ satisfies (25) for $j = 1$ with a unique value of Θ. Consider $\Theta^{-1}w(s,\Theta)$ for $\Theta > 0$. It can be shown as with Theorem 3 that $\Theta^{-1}w(s,\Theta) \to \bar{w}(s)$ as $\Theta \to \infty$ for all $s \in S$, where $\bar{w}(s)$ is the solution to

$$\bar{w}'(s) = -\min_{a_1 \in A_s^1} \max_{a_2 \in A_s^2} \{d(s,a_1,a_2)^{-1}[b(s,a_1,a_2)\bar{w}(s) - 1]\}$$

satisfying

$$\theta_0 \int_S \int_{r_0}^{s} \bar{w}(y)dy dL_0(s) + \tau_0 \bar{w}(r_0) - \sigma_0 = 0 .$$

After showing that

$$\theta_1 \int_S \int_{r_0}^{s} \bar{w}(y)\,dy\,d\mu_1(s) - \pi_1 \bar{w}(r_1) - \sigma_1 < 0$$

we conclude that the left hand side of (25) for $j = 1$ with $w(s,\Theta)$ substituted for $w(s)$ diverges to $\mp\infty$ as $\Theta \to \pm\infty$. By continuity, $w(s,\Theta)$ satisfies (25) for $j = 1$ with some value Θ. This solution Θ is unique; otherwise a contradiction can be derived as was done with Theorem 3 to show the unicity of $v(r_0)$.

<u>Theorem 17</u>. With undiscounted costs and conservative boundary conditions, let Θ be the unique number such that (24) has a solution $w(s)$ satisfying (25). A control $a = (a_1,a_2) \in M$ is optimal if and only if

(26) $$\max_{a_2 \in A_s^2} \{d(s,a_1(s),a_2)^{-1}[b(s,a_1(s),a_2)w(s) - \Theta + c(s,a_1(s),a_2)]\}$$

$$= d(s,a_1(s),a_2(s))^{-1}[b(s,a_1(s),a_2(s))w(s) - \Theta + c(s,a_1(s),a_2(s))]$$

$$= \min_{a_1 \in A_s^1} \{d(s,a_1,a_2(s))^{-1}[b(s,a_1,a_2(s))w(s) - \Theta + c(s,a_1,a_2(s))]\}$$

for every $s \in S$ which is a continuity point of $a(s)$, and, for $j = 0$ and $j = 1$, σ_j 0 implies

(27) $$\max_{a_2 \in A_{r_j}^2} c(r_j,a_1(r_j),a_2) = c(r_j,a_1(r_j),a_2(r_j))$$

$$= \min_{a_1 \in A_{r_j}^1} c(r_j,a_1,a_2(r_j)).$$

Moreover, if $a(s)$ is optimal then $\theta = \theta(a_1, a_2)$ is the value of the game.

This proof is essentially identical to that for Theorem 11, so it will be omitted. A diffusion process zero sum, two-person game problem in the undiscounted cost, conservative process case can, in principle, be solved in the same manner as with the discounted cost case. Moreover, there exist sufficient conditions analogous to those of Theorem 12 and Corollary 13 for equations (26) and (27) to be satisfied by some Borel measurable control $a(s)$.

Example.

$$S = [0,1] \qquad d(s,a_1,a_2) = A > 0$$

$$A_s^1 = [-Y,Y] \subset E, \quad Y > 0 \qquad b(s,a_1,a_2) = a_1 a_2$$

$$A_s^2 = [-Z,Z] \subset E, \quad Z > 0 \qquad c(s,a_1,a_2) = Cs$$

Suppose both boundary conditions are pure reflection. For any value of s, $w(s)$, and θ we have

$$\min_{a_1 \in A_s^1} \max_{a_2 \in A_s^2} \{A^{-1}[a_1 a_2 w(s) - \theta + Cs]\}$$

$$= \max_{a_2 \in A_s^2} \min_{a_1 \in A_s^1} \{A^{-1}[a_1 a_2 w(s) - \theta + Cs]\} = A^{-1}[Cs - \theta] ,$$

so $a(s) = (0,0)$ is the optimal control for all $s \in S$. Therefore, the value of this game is given by (3), (4) to be $\Theta = C/2$, with $w(s) = (\Theta/A)s - \frac{1}{2}(C/A)s^2$.

5. Application: Optimal Welfare Policies.

Suppose the problem of determining some government's optimal welfare policy can be posed as a diffusion process zero sum, two-person game as follows. Let the state space correspond to some population so that the state of the process will equal the number of people receiving welfare. Assume the boundary conditions are pure reflection. Let the first control component, operated by the government, be the cost of welfare per person per unit time. Let the second control component, operated by the population, equal the cost of civil disturbances per person per unit time. Finally, let the costs of this welfare game be represented by a continuous movement cost which equals the sum of the total welfare and total civil disturbance costs per unit time. We naively assume the civil unrest cost to the government equals the reward (e.g., satisfaction) to the participants. Thus, the total cost to the government equals the total reward to the population, and the government acts to minimize, while the population acts to maximize, the expected costs of this game.

Presumably, the drift and diffusion coefficients should reflect the fact that the greater the welfare cost per person the greater the tendency for the number of people receiving welfare to increase

Similarly, these coefficients should reflect any tendency for the number of people receiving welfare to decrease as the civil disturbance cost is increased. This tendency would exist, for example, if the government were to retaliate by more strictly enforcing welfare elgibility requirements. In summary, for every combination of controls, the number of people receiving welfare can be represented by a conservative diffusion process.

<u>Example</u>. This example involves undiscounted costs. Let $S = [0,P]$ and suppose $A_s^i = [0,C_i]$ for $i = 1,2$. Let the diffusion coefficient be $A > 0$, the drift coefficient be the general function $b(s,a_1,a_2)$, and the continuous movement cost $a_1 s + a_2 P$. By inspection, if $a(s) = (0,C_2)$ and $\Theta = C_2 P$, then (3), (4) have the unique solution $w(s) = 0$. Moreover, this control $a(s)$ satisfies (26) so it is optimal and Θ is the value of the game.

6. <u>The Non-zero Sum, N-person Game Problem</u>.

The remainder of this chapter describes a class of controlled diffusion processes whose control problems can be viewed as non-zero sum, N-person games. We consider the multi-person controlled diffusion process of Section 1; these processes are controlled by N persons and generate N streams of costs $(N \geq 2)$. Controller i $(i = 1,\cdots,N)$, who operates the i^{th} control, endeavors to choose a control $a_i \in M_i$ so as to minimize the costs of the i^{th} cost stream generated by the process. A game situation exists by virtue of the fact that the cost to the i^{th} person is influenced by the actions of the other players.

The optimality criterion used for these processes is that of a Nash equilibrium point. If an admissible expected cost is defined to be the expected cost corresponding to some admissible control, then the solution to this game will be some admissible control whose corresponding expected cost is a Nash equilibrium point with respect to all admissible costs. Thus if player i unilaterally deviates from his component of this optimal control, then his expected costs will either be unchanged or increased. The adoption of the Nash equilibrium point optimality criterion is made in recognition of the fact that a variety of meritorius optimality criteria exist for non-zero sum, N-person game problems. In particular, a "prisoner's dilemma" situation might exist where the players would gain by deviating from the Nash equilibrium point solution in a cooperative manner.

The following two sections provide results respectively for the discounted cost case and the undiscounted cost case. The main result of each section is a necessary and sufficient condition for a control to be optimal. In addition, a method based upon the theory of differential games is provided for solving a diffusion process non-zero sum, N-person game problem. This method is substantially the same as a method used for solving an ordinary diffusion process optimal control problem. The optimizing solution of an equation is substituted into a differential equation whose solution, in turn, is used to obtain the optimal control. The final two sections indicate two possible applications of this model: control of pollution and optimal warfare strategies.

To minimize ambiguity, the following terminology is used. The

i^{th} control $a_i \in M_i$ is operated by the i^{th} player and is generally a vector-valued function on S. The control $a = (a_1, \ldots, a_N)$ is the vector consisting of the N players' controls.

7 The Non-zero Sum Problem with Discounted Costs

Let $v(s,a) = v(s,a_1, \ldots, a_N)$ denote the expected discounted cost of a process corresponding to the admissible control $a \in M$, and let $v_i(s,a)$ be its i^{th} component, $i = 1, \ldots, N$. Then $v(s,a)$ will be the unique solution of (1), (2). The control $\hat{a} \in M$ is said to be optimal, that is, a solution of the game if it is a Nash equilibrium point of the expected discounted cost functions, that is,

$$v_i(s,\hat{a}) \le v_i(s,\hat{a}_1, \ldots, \hat{a}_{i-1}, a_i, \hat{a}_{i+1}, \ldots, \hat{a}_N)$$

for all $s \in S$, all $a_i \in M_i$, and each $i = 1, \ldots, N$. In this case $v(s,\hat{a})$ is said to be a value of the game.

To simplify our notation define

$$\operatorname*{val}_{z \in Z} g(z) = \{g(\hat{z}) : \hat{z} \text{ is a Nash equilibrium point of } g \text{ on } Z\}$$

where $Z_i \subset E$ for $i = 1, \ldots, N$, $Z = Z_1 \times \cdots \times Z_N$, and the function $g : Z \to E^N$. The main result of this section is the following.

Theorem 18. A control $\hat{a} \in M$ is optimal if and only if for each $s \in S$

which is a continuity point of $\hat{a}(s)$

$$(28) \qquad d(s,\hat{a}(s))^{-1}[b(s,\hat{a}(s))v'(s) - \lambda v(s) + c(s,\hat{a}(s))]$$

$$\varepsilon \operatorname*{val}_{a\varepsilon A_s} \{d(s,a)^{-1}[b(s,a)v'(s) - \lambda v(s) + c(s,a)]\} ;$$

where $v(s) = v(s,\hat{a})$, and

$$(29) \qquad \sigma_j(c(r_j,\hat{a}(r_j)) - \gamma_j) = 0 \qquad j = 0,1,$$

where $\gamma_j \varepsilon \operatorname*{val}_{a\varepsilon A_{r_j}} c(r_j,a)$, $\qquad j = 0,1$

Proof. Let $\hat{a}$ be optimal. For arbitrary i let $\hat{a}_1,\ldots,\hat{a}_{i-1},\hat{a}_{i+1},\ldots,\hat{a}_N$ be fixed so that $v_i(s,\hat{a}) = \inf_{a_i \varepsilon M_i} v_i(s,\hat{a}_1,\ldots,\hat{a}_{i-1},a_i,\hat{a}_{i+1},\ldots,\hat{a}_N)$ is the minimal expected discounted cost of an optimal control problem and $\hat{a}_i$ is one of its optimal controls. By Theorem 6 of Chapter II we have

$$(30) \qquad d(s,\hat{a}(s))^{-1}[b(s,\hat{a}(s))v_i'(s) - \lambda v_i(s) + c_i(s,\hat{a}(s))]$$

$$= \min_{a_i \varepsilon A_s^i}\{d(s,\hat{a}_1(s),\ldots,a_i,\ldots,\hat{a}_N(s))^{-1}[b(s,\hat{a}_1(s),\ldots,a_i,$$

$$\ldots,\hat{a}_N(s))v_i'(s) - \lambda v_i(s) + c_i(s,\hat{a}_1(s),\ldots,a_i,\ldots,\hat{a}_N(s))]\}$$

for each $s \varepsilon S$ which is a continuity point of $\hat{a}_i(s)$ and

(31) $$\sigma_j(c_i(r_j,\hat{a}(r_j)) - \gamma_{ij}) = 0 , \qquad j = 0,1,$$

where $\gamma_{ij} = \min_{a_i \epsilon A^i_{r_j}} c_i(r_j,\hat{a}_1(r_j),\cdots,a_i,\cdots,\hat{a}_N(r_j)),\ j = 0,1$

Since i is arbitrary, (28) and (29) must be true.

Conversely, suppose (28) and (29) hold and let i be arbitrary. Now (30) and (31) must hold, so by Theorem 6 of Chapter II we see that $v_i(s,\hat{a})$ is the minimal expected discounted cost of the ordinary optimal control problem: minimize $v_i(s,\hat{a}_1,\cdots,a_i,\cdots,\hat{a}_N)$ subject to $a_i \epsilon M_i$. Since i is arbitrary, $\hat{a}$ defines a Nash equilibrium point for this game.

Theorem 18 is substantially different from Theorem 11 for the zero sum, two-person game situation in one respect. In each case the necessary and sufficient condition is a function of the solution to a differential equation. With Theorem 18, this solution is explicitly a function of some control $a \epsilon M$, whereas in the case of Theorem 11 the corresponding differential equation solution is explicitly independent of any control $a \epsilon M$. Thus, given a control $a \epsilon M$, one can determine $v(s,a)$ with Theorem 1 and then ascertain whether $a(s)$ is optimal with Theorem 18. Conversely, an optimal control $a(s)$ will satisfy (28) and (29). However, Theorem 18 does not provide an explicit procedure for solving the diffusion process non-zero sum, N-person game problem.

The following computational procedure is based upon a method devised by Starr and Ho [21] as well as Case [3] for solving non-zero sum differential games. Let $t = (t_1, \cdots, t_N)$ and $u = (u_1, \cdots, u_N)$ and define $g(s,t,u,a) : S \times E^{2N} \times K \to E^N$ by

$$g(s,t,u,a) \equiv d(s,a)^{-1}[b(s,a)u - At + c(s,a)] .$$

Now consider the point-to-set map $\Gamma : S \times E^{2N} \to K$ defined by

$$\Gamma(s,t,u) = \{a \in A_s | g(s,t,u,a) \in \operatorname*{val}_{a \in A_s} g(s,t,u,a)\} .$$

If $\sigma_j > 0$ for $j = 0$ or $j = 1$, then redefine $\Gamma(r_j,t,u)$ so that $\Gamma(r_j,t,u) = \{a \in A_{r_j} | c(r_j,t,u,a) \in \operatorname*{val}_{a \in A_{r_j}} c(r_j,t,u,a)\}$. If $\Gamma(s,t,u) \neq \phi$ for each (s,t,u), then choose a function $a(s,t,u)$ with $a(s,t,u) \in \Gamma(s,t,u)$ for each (s,t,u), substitute $a(s,v(s),v'(s))$ for $a(s)$ in (1) and (2), and solve for $v(s)$. If $v(s)$ exists, then it is a value of the game. If $a(s) = a(s,v(s),v'(s))$ is piecewise continuous, then it is an optimal control and $v(s) = v(s,a)$.

Note that this procedure may break down in three different ways: $\Gamma(s,t,u)$ may not exist; $v(s)$ may not exist; and $a(s,v(s),v'(s))$ may not be piecewise continuous. The reason why $v(s)$ may fail to exist, although $a(s,t,u)$ does, is that the Euclidean norm of $g(s,t,u,a(s,t,u))$ may fail to be continuous on $S \times E^{2N}$. Since most differential equation theory existence theorems specify some form of continuity requirement, counterexamples can be easily constructed. The following proposition serves to characterize Γ.

Proposition 19. If A_s is continuous on S, then Γ is a closed map on $S \times E^{2N}$.

Proof. For each $i = 1,\cdots,N$, define the point-to-set map $\Gamma_i : S \times E^{2N} \times K \to K_i$ by

$$\Gamma_i(s,t,u,a) = \arg\min_{a_i \varepsilon A_s^i} g_i(s,t,u,a) \; .$$

The continuity of $g_i(s,t,u,a)$ implies Γ_i is a closed map, so its graph $D_i = \{(s,t,u,a) | a_i \varepsilon \Gamma_i(s,t,u,a)\}$ is a closed set. The map Γ is thus closed because its graph $D_1 \cap \cdots \cap D_N$ is closed.

Proposition 20. Suppose each component i of $g(s,t,u,a)$ is quasi-convex in $a_i \varepsilon K_i$ for each $s \varepsilon S$, each $a_j \varepsilon K_j$ $(j = 1,\cdots,i-1,i+1,\cdots,N)$, and each $t, u \varepsilon E^N$, and assume A_s^i is convex for each $s \varepsilon S$ and $i = 1,\cdots,N$. Then $\Gamma(s,t,u) \neq \phi$ for each $(s,t,u) \varepsilon S \times E^{2N}$.

The proof of Proposition 20 is omitted because it follows easily from Rosen [17] and Sion [20]. If Proposition 20 holds and the Nash equilibrium point is unique for each (s,t,u), then the closed map $\Gamma(s,t,u)$ is simply a piecewise continuous function. This observation leads to the following existence theorem. Following Rosen [17], the function $g : E^N \to E^N$ is said to be diagonally strictly convex for $a \varepsilon K$ if for each $a^0, a^1 \varepsilon K$ we have

$$(a^1 - a^0)^T f(a^0) + (a^0 - a^1)^T f(a') < 0 ,$$

where $f(a) = \left(\frac{\partial g_1}{\partial a_1}, \cdots, \frac{\partial g_N}{\partial a_N}\right)^T$. A sufficient condition that $g(a)$ be be diagonally strictly convex is that the symmetric matrix $[F(a) + F^T(a)]$ be positive definite for $a \in K$, where $F(a)$ is the Jacobian with respect to a of $f(a)$ (Rosen [17]).

Theorem 21. Assume A_s is continuous on S with A_s^i convex for each $s \in S$ and $i = 1, \cdots, N$. Also assume $d(s,a)$ is constant in a, $b(s,a)$ is affine in a, and $c(s,a)$ is diagonally strictly convex in a, all for each $s \in S$. Then a solution exists for this diffusion process game.

Proof. The function $g(s,t,u,a)$ is strictly diagonally convex, so by Rosen [17] there exists a unique Nash equilibrium point for each (s,t,u), that is, by the above remarks $\Gamma(s,t,u)$ is a continuous function on $S \times E^{2N}$. By differential equation theory and the arguments of Section 3, there exists a solution $v(s)$ to (1), (2) with $\Gamma(s,v(s),v'(s))$ substituted for $a(s)$. Hence a solution of the game is $a(s) = \Gamma(s,v(s),v'(s)) \in M$, and the corresponding value of the game is $v(s) = v(s,a)$.

Example. This example is a two-person game. Let the state space and sets of admissible control values equal the unit interval. Let $d(s,a_1,a_2) = 1$, $b(s,a_1,a_2) = a_1 + a_2$, and $c_i(s,a_1,a_2) = C + a_i$, $i = 1,2$, where C is a constant. Suppose the boundary condition at

r_0 is reflection and that absorption occurs at r_1 with cost λ_1. Following the above procedure, we have for $i = 1,2$ that $a_i(s,t,u) = 1$ for $u_i \leq 1$ and $a_i(s,t,u) = 0$ otherwise. By symmetry we have that $v_1(s) = v_2(s)$ is the solution to

$$(32) \qquad v_1''(s) = -2a_1(s,v(s),v'(s))v_1'(s) + \lambda v_1(s) - K - a_1(s,v(s),v'(s))$$

satisfying $v'(r_0) = 0$ and $v(r_1) = \lambda_1$. In some neighborhood of r_0 we have $v_1'(s) > -1$, so in this neighborhood $a(s,v(s),v'(s)) = (0,0)$ and, for some constant q, $v(s) = qe^{\sqrt{\lambda}s} + qe^{-\sqrt{\lambda}s} + C/\lambda$. If

$$\lambda_1 \geq \frac{e^{\sqrt{\lambda}} + e^{-\sqrt{\lambda}}}{\sqrt{\lambda}\left(e^{-\sqrt{\lambda}} - e^{\sqrt{\lambda}}\right)} + \frac{C}{\lambda},$$ then q can be chosen so that $v_1(1) = \lambda_1$

and $v_1'(s) \geq -1$ for all $s \in S$, in which case $a(s) = (0,0)$ is optimal

for all $s \in S$. If $\lambda_1 < \dfrac{e^{\sqrt{\lambda}} - e^{-\sqrt{\lambda}}}{\sqrt{\lambda}\left(e^{-\sqrt{\lambda}} - e^{\sqrt{\lambda}}\right)} + \dfrac{C}{\lambda}$, then for

some $s_0 \in (0,1)$ we have $v_1'(s_0) = -1$ and $a(s) = (0,0)$ optimal for all $s \in [0,s_0)$. For $v_1(s)$ to exist, we must have $v_1''(s_0) < 0$ when $a_1(s_0,v(s_0),v'(s_0)) = 1$ in (32). But this is easily verified, so for all $s \geq s_0$ in some neighborhood of s_0 we have $a(s) = (1,1)$ optimal and $v(s) = t_1e^{u_1s} + t_2e^{u_2s} + (C+1)/\lambda$, where $u_1 = -1 + \sqrt{1+\lambda}$, $u_2 = -1 - \sqrt{1+\lambda}$, and t_1 and t_2 are constants. It remains to show $a(s) = (1,1)$ is optimal for all $s \geq s_0$. Suppose not, but that $s_1 < 1$, say, is the smallest $s > s_0$ such that $v_1'(s_1) = -1$. Then $v_1''(s_1) < v_1''(s_0) < 0$, a contradiction. The unknown constants q, t_1, t_2, and s_0 can be solved from the boundary conditions and the fact that $v'(s)$ is continuous with $v'(s_0) = -1$.

8. The Non-zero Sum Problem with Undisounted Costs

The non-zero sum, N-person diffusion process game problem with undiscounted costs will be one of two types, depending on whether the boundary conditions are conservative or non-conservative. The results in this section parallel those of Sections 4 and 7, and, consequently, they will be brief. The conservative case will be treated in the second half of this section.

For the purposes of this section, the boundary conditions are said to be non-conservative if at least one boundary is absorbing and neither boundary is purely adhesive, that is,

$$\kappa_0 + \kappa_1 > 0, \qquad \kappa_j + \pi_j + \varepsilon_j > 0, \qquad j = 0,1$$

Let $v(s,a) = v(s,a_1,\ldots,a_N) = v(s)$ denote the expected undiscounted cost of such a process corresponding to the admissible control $a \in M$. Then $v(s,a)$ will be the unique solution of (1), (2) with $\lambda = 0$. The control $a \in M$ is said to be optimal if it defines a Nash equilibrium point with respect to the expected cost functions, that is,

$$v_i(s,a) \le v_i(s,a_1,\ldots,a_{i-1},a_i,a_{i+1},\ldots,a_N)$$

for all $s \in S$, all $a_i \in M_i$, and each $i = 1,\cdots,N$. In this case $v(s,\hat{a})$ is said to be a value of the game.

Theorem 22. With non-conservative boundary conditions, a control $a \in M$ is optimal if and only if for each $s \in S$ which is a continuity point of $a(s)$

$$d(s,a(s))^{-1}[b(s,a(s))v'(s) + c(s,a(s))]$$

$$\in \operatorname*{val}_{a \in A_s} \{d(s,a)^{-1}[b(s,a)v'(s) + c(s,a)]\},$$

where $v(s) = v(s,a)$, and

$$\sigma_j(c(r_j,a(r_j)) - \gamma_j) = 0, \qquad j = 0,1,$$

where $\gamma_j \in \operatorname*{val}_{a \in A_{r_j}} c(r_j,a)$, $j = 0,1$.

The proof is essentially the same as that for Theorem 18, so it will be omitted. Moreover, the remarks and computational procedure that follow Theorem 18 apply to this case as well

Example. This example is identical to that of the preceding section except that the costs are undiscounted Proceeding in a similar manner, we have that $v_1(s) = v_2(s)$ is the solution to

$$v_1''(s) = -2a_1(s,v(s),v'(s))v_1'(s) - C - a_1(s,v(s),v'(s))$$

satisfying $v_1'(r_0) = 0$ and $v_1(r_1) = \lambda_1$, and that $a_1(s,v(s),v'(s)) = a_2(s,v(s),v_1'(s)) = 1$ $(= 0)$ if $v'(s) \leq -1$ (≥ -1). If $C \leq 1$, then $a(s) = (0,0)$ is optimal for all $s \in S$ and $v_1(s) = \lambda_1 + C(1 - s^2)/2$. If $C > 1$, then $a(s) = (0,0)$ is optimal and $v_1(s) = \lambda_1 + C/2 + (C - 1)(\exp(-2 + 2/C) - 1)/4 - Cs^2/2$ on $[0,1/C)$, and $a(s) = (1,1)$ is optimal and $v_1(s) = \lambda_1 + (C + 1)(1 - s)/2 + \exp(-2 + 2/C)(1 - \exp(2 - 2s)(C - 1)/4$ on $(1/C,1]$.

We now discuss the other type of undiscounted cost problem, the conservative case. For purposes of this section, the boundary conditions are said to be conservative if neither boundary is absorbing and at least one boundary is not purely adhesive, that is,

$$\kappa_0 + \kappa_1 = 0, \qquad \pi_0 + \delta_0 + \pi_1 + \delta_1 > 0.$$

Let $\Theta(a) = \Theta(a_1, \ldots, a_N) = \Theta$ denote the vector of mean costs per unit time of such a process corresponding to the control $a \in M$. Then $\Theta(a)$ is the unique vector to which there exists a solution $w(s,a)$ to (3) and (4). The control $a \in M$ is said to be optimal if it defines a Nash equilibrium point with respect to the mean costs, that is,

$$\Theta_i(a) \leq \Theta_i(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_N)$$

for all $a_i \in M_i$, $i = 1, \ldots, N$. In this case $\Theta(a)$ is said to be a value of the game.

Theorem 23 With conservative boundary conditions a control $a \in M$ is optimal if and only if for each $s \in S$ which is a continuity point of $a(s)$

$$d(s,a(s))^{-1}[b(s,a(s))w(s) - \Theta + c(s,a(s))]$$

$$\in \operatorname*{val}_{a \in A_s} \{d(s,a)^{-1}[b(s,a)w(s) - \Theta + c(s,a)]\},$$

where $w(s) = w(s,a)$ and $\Theta = \Theta(a)$, and

$$\sigma_j(c(r_j,a(r_j)) - \gamma_j) = 0, \qquad j = 0,1,$$

where $\gamma_j \in \operatorname*{val}_{a \in A_{r_j}} c(r_j,a)$, $j = 0,1$.

The proof is essentially the same as that for Theorem 18, so it will be omitted. Moreover, the remarks and computational procedure that follow Theorem 18 apply to this case as well.

Example This is an example of an N-person game. Let $S = A_s^i = [-1,1]$ for $i = 1,\cdots,N$ and all $s \in S$, $d(s,a) = 1$, $b(s,a) = a_1 + \cdots + a_N$, and $c_i(s,a) = |s|$ for $i = 1,\cdots,N$. Suppose reflection occurs at each boundary. The i^{th} control $a_i(s) = -1$ if $w_i(s) \geq 0$ and $a_i(s) = 1$ otherwise. By symmetry, $w_1(s) = \cdots = w_N(s)$ so $w_1(s)$ is the solution to

$$w_1'(s) = \begin{cases} -Nw_1(s) - c + |s| , & w_1(s) \geq 0 \\ Nw_1(s) - c + |s| , & w_1(s) \leq 0 \end{cases}$$

satisfying $w_1(1) = w_1(-1) = 0$. By symmetry we must have $w_1(0) = 0$, so we verify that $c = \frac{1}{1 - e^{-N}} - \frac{1}{N}$ and that for $i = 1, \ldots, N$

$$a_i(s) = \begin{cases} 1 , & s \in [-1,0) \\ -1 , & s \in (0,1] . \end{cases}$$

9. Application: Control of Pollution

Suppose that the index of pollution is constrained to fall between zero and some positive number. This would be the case, for example, when dealing with an air basin or a body of water. Assume that a collection of N factories, automobiles, or similar polluting mechanisms contributes to this pollution and that each such mechanism can control this index of pollution by choosing the amount of its waste products that is emitted as a pollutant as opposed to being processed in a pollution-free manner. Finally, assume that there exists a cost to each controller for each level of control as well as to each value of the pollution index. Then a non-zero sum, N-person diffusion process game may perhaps be used as a model of this pollution system.

This pollution model is a generalization of one in Chapter II.

The state of the process will correspond to the index of pollution, and the boundary behavior, at zero and the maximum index value, will be reflection or possibly reflection combined with adhesion. An admissible control component $a_i(s)$ will be a piecewise continuous function of the state space that will represent the fraction of the i^{th} controller's waste products that is being emitted as a pollutant. Presumably, the bigger the fraction of wastes being emitted, the smaller the control cost rate but the bigger the drift coefficient. Similarly, the bigger the pollution index, the greater the pollutant cost rate. An optimal control will be an admissible control which yields a Nash equilibrium point with respect to the expected costs.

<u>Example</u>. This example involves undiscounted costs and N polluting mechanisms. Let $S = A_s^i = [0,1]$ for $i = 1, \ldots, N$ so that the i^{th} control component equals the fraction of the i^{th} polluting mechanism's wastes that is being processed in a pollution-free manner. Let $d(s,a) = 1$, $b(s,a) = -a_1 - \cdots - a_N$, and $c_i(s,a) = Cs + a_i$ for $i = 1, \ldots, N$, and suppose pure reflection occurs at each boundary. By Theorem 23 we have

$$a_i(s) = \begin{cases} 0, & w_i(s) \geq 1 \\ 1, & w_i(s) \leq 1, \end{cases}$$

and by symmetry $w_1(s) = \cdots = w_N(s)$. If $a_i(s) = 0$ for all $s \in S$, then $\theta_i = C/2$ and $w_i(s) = \theta_i s - Cs^2/2$, so this control is optimal if $C \leq 8$. On the other hand, suppose $C > 8$ so that there exist

$0 < s_0 < s_1 < 1$ such that $w_i(s_0) = w_i(s_1) = 1$ and $a_i(s) = 1$ for all $s \in (s_0, s_1)$. On $[s_0, s_1]$ we must have

$$w_i(s) = te^{Ns} + \frac{Cs + 1 - C_1}{N} + \frac{C}{N^2}$$

where t is determined from $w_i(s_0) = w_i(s_1)$ to be

$$t = \frac{C(s_0 - s_1)}{N\left(e^{Ns_1} - e^{Ns_0}\right)} < 0 .$$

Since $w_i(s_0) = 1$, we have

$$\frac{C(s_0 - s_1)e^{Ns_0}}{N\left(e^{Ns_1} - e^{Ns_0}\right)} + \frac{Cs_0 + 1 - C_1}{N} + \frac{C}{N} = 1 ,$$

but for large enough N this equation will not be satisfied by any $C_1 \geq 0$. Hence if $C > 8$, an optimal control may not exist.

10. Application: Optimal Warfare Strategies

Suppose a war between two antagonists is characterized by an index that varies continuously between two real numbers $r_0 < r_1$, and this war is terminated in favor of the first (second) antagonist when this index first attains $r_1(r_0)$. For example, this index could represent the portion of some land mass under the control of the first antagonist as opposed to being under the control of the second. Or this index could represent the portion of some population that is allegiant to one government as opposed to being allegiant to a second. Suppose each antagonist can control this index by choosing alternative levels of fighting effort. Finally, suppose costs to each antagonist are associated with each of the two possible outcomes as well as with alternative levels of fighting effort and the index value. Then the problem of determinig the optimal level of fighting for these two antagonists can perhaps be resolved by consideration of a non-zero sum, two-person diffusion process game.

The state of the diffusion process will correspond to the warfare index, and the boundary behavior, at r_0 and r_1, will be absorption or possible absorption combined with another type of boundary phenomenon. Let the termination costs at each boundary be positive for the loser and negative for the winner. Let the control represent levels of fighting effort for the two players so that each player's continuous movement cost represents the cost to him of the fighting levels and warfare index being at particular values for one unit of time. If a termination at boundary r_1 represents victory for player one, then presumably the bigger the first (second) player's control the greater the tendency for the warfare index to increase (decrease). Similarly, the greater the level of fighting the greater the continuous movement cost.

Example. Let $S = A_s^i = [01]$ and suppose $d(s,a) = 1$, $b(s,a) = a_1 - a_2$, and $c_i(s,a) = a_i$ for $i = 1,2$. Assume the boundary behavior is absorption and that termination at r_j represents defeat for player

$j+1$, where $j = 0,1$, with termination cost $C > 0$ for the loser and $-C$ for the winner. Finally, suppose the costs are undiscounted, so we consider Theorem 22.

By symmetry we must have $a_1(s) = a_2(1-s)$ and $v_1(s) = v_2(1-s)$ for all $s \in S$. Moreover, $a_1(s) = 0$ if $v_1'(s) \geq -1$ and $a_1(s) = 1$ otherwise. If $C \leq \frac{1}{2}$ and $a(s) = (0,0)$, then $v_1(s) = -2Cs + C$ so $v_1'(s) \geq -1$ and $a(s) = (0,0)$ is optimal. If $C \geq \frac{3}{4}$ and $a(s) = (1,1)$, then $v_1(s) = \frac{-s^2}{s} + (-2C + \frac{1}{2})s + C$ so $v_1'(s) \leq -1$ and $a(s) = (1,1)$ is optimal. Finally, if $\frac{1}{2} < C < \frac{3}{4}$, then the following argument will show that for some $s_0 \in (0,\frac{1}{2})$ where $v_1'(s_0) = -1$ we have $a_1(s) = 0$ optimal on $[0,s_0)$ and $a_1(s) = 1$ optimal on $(s_0,1]$. On $[0,s_0)$ we have $a(s) = (0,1)$, so $v_1(s) = C + e^{-s_0} - e^{s-s_0}$. On $(s_0, 1-s_0)$ we have $a(s) = (1,1)$, so

$$v_1(s) = \frac{1}{2}(s - s_0)^2 + C + e^{-s_0} + s_0 - 1 - s$$

On $(1-s_0, 1)$ we have $a(s) = (1,0)$, so

$$v_1(s) = 5s_0 - \frac{5}{2} - 2s_0^2 + C + e^{-s_0} + (1 - 2s_0)e^{1-s_0-s} - s$$

Soving the equation $v_1(1) = -C$ yields a unique soltuion for $s_0 \in (0,\frac{1}{2})$ if $\frac{1}{2} < C < \frac{3}{4}$, so we are done provided $a(s)$ is optimal. The function $v_1(s)$ is concave on $[0,1-s_0)$ and convex on $(1-s_0,1]$, so $v_1'(s) \geq -1$ on $[0,s_0)$ and $v_1'(s) \leq -1$ on $(s_0,1]$ provided $v_1'(1) \leq -1$. This last inequality is true, so $a(s)$ is optimal.

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